The Open University

MST124

Essential mathematics 1

Handbook

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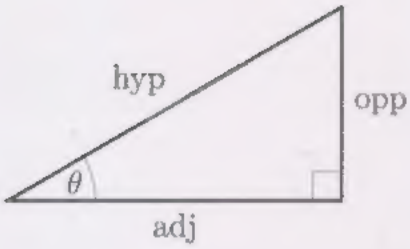
Contents

Quick reference material

Trigonometric ratios

Degrees and radians

$360° = 2\pi$ radians $\qquad 1° = \frac{\pi}{180}$ radians $\qquad 1$ radian $= \left(\frac{180}{\pi}\right)^{\circ}$

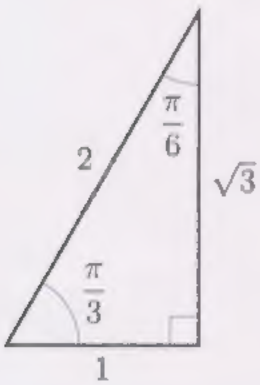


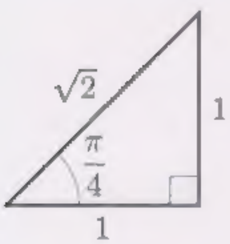
Sine, cosine and tangent

For an acute angle θ,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

Mnemonic: SOH CAH TOA.





Special angles

θ in radians	θ in degrees	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	$0°$	0	1	0
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	$90°$	1	0	–
π	$180°$	0	-1	0

Note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and that $\tan\left(\frac{\pi}{2}\right)$ is undefined.

Trigonometric identities

tan, cosec, sec, cot in terms of sin and cos

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad$ $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ $\quad$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean identities

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Symmetry identities

$\sin(-\theta) = -\sin\theta$ $\quad$ $\sin(\theta + 2\pi) = \sin\theta$ $\quad$ $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos(-\theta) = \cos\theta$ $\quad$ $\cos(\theta + 2\pi) = \cos\theta$ $\quad$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan(-\theta) = -\tan\theta$ $\quad$ $\tan(\theta + \pi) = \tan\theta$

Angle sum and angle difference identities

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Double-angle and half-angle identities

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= 1 - 2\sin^2\theta \qquad \text{so} \quad \sin^2\theta = \tfrac{1}{2}(1 - \cos(2\theta))\\ &= 2\cos^2\theta - 1 \qquad \text{so} \quad \cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta))\end{aligned}$$

$\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Sum to product identities

$\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

$\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

$\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

$\cos A - \cos B = -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

Product to sum identities

$\sin A\sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$

$\cos A\cos B = \frac{1}{2}\cos(A + B) + \frac{1}{2}\cos(A - B)$

$\sin A\cos B = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(A - B)$

Index laws and logarithm laws

Index laws

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

$$(ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^{1/n} = \sqrt[n]{a} \qquad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Logarithm laws

$y = \log_b x$ is equivalent to $x = b^y$

$$\log_b 1 = 0 \qquad \log_b(b^x) = x$$

$$\log_b b = 1 \qquad b^{\log_b x} = x$$

$$\log_b x + \log_b y = \log_b(xy)$$

$$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$$

$$r\log_b x = \log_b(x^r)$$

Logarithm laws for natural logarithms

$y = \ln x$ is equivalent to $x = e^y$

$$\ln 1 = 0 \qquad \ln e^x = x$$

$$\ln e = 1 \qquad e^{\ln x} = x$$

$$\ln x + \ln y = \ln(xy)$$

$$\ln x - \ln y = \ln\left(\frac{x}{y}\right)$$

$$r\ln x = \ln(x^r)$$

Standard derivatives and standard integrals

Standard derivatives

Function	Derivative
a (constant)	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$

Standard indefinite integrals

Function	Indefinite integral
a (constant)	$ax + c$
x^n $(n \neq -1)$	$\dfrac{1}{n+1}x^{n+1} + c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + c$ or $\ln x + c$, for $x > 0$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\operatorname{cosec}^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$ or $-\cos^{-1} x + c$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + c$

Standard series

Standard Taylor series

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots \qquad \text{for } x \in \mathbb{R}$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \qquad \text{for } x \in \mathbb{R}$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \text{for } -1 < x < 1$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

(α can be any real number) for $-1 < x < 1$

Arithmetic series

The finite arithmetic series with first term a, common difference d and n terms has sum

$$a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \tfrac{1}{2}n\,(2a+(n-1)d).$$

If the last term of the series is l, the sum is $\frac{1}{2}n(a+l)$.

Geometric series

The finite geometric series with first term a, common ratio $r \neq 1$ and n terms has sum

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

The infinite geometric series with first term a and common ratio r, with $-1 < r < 1$, has sum

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

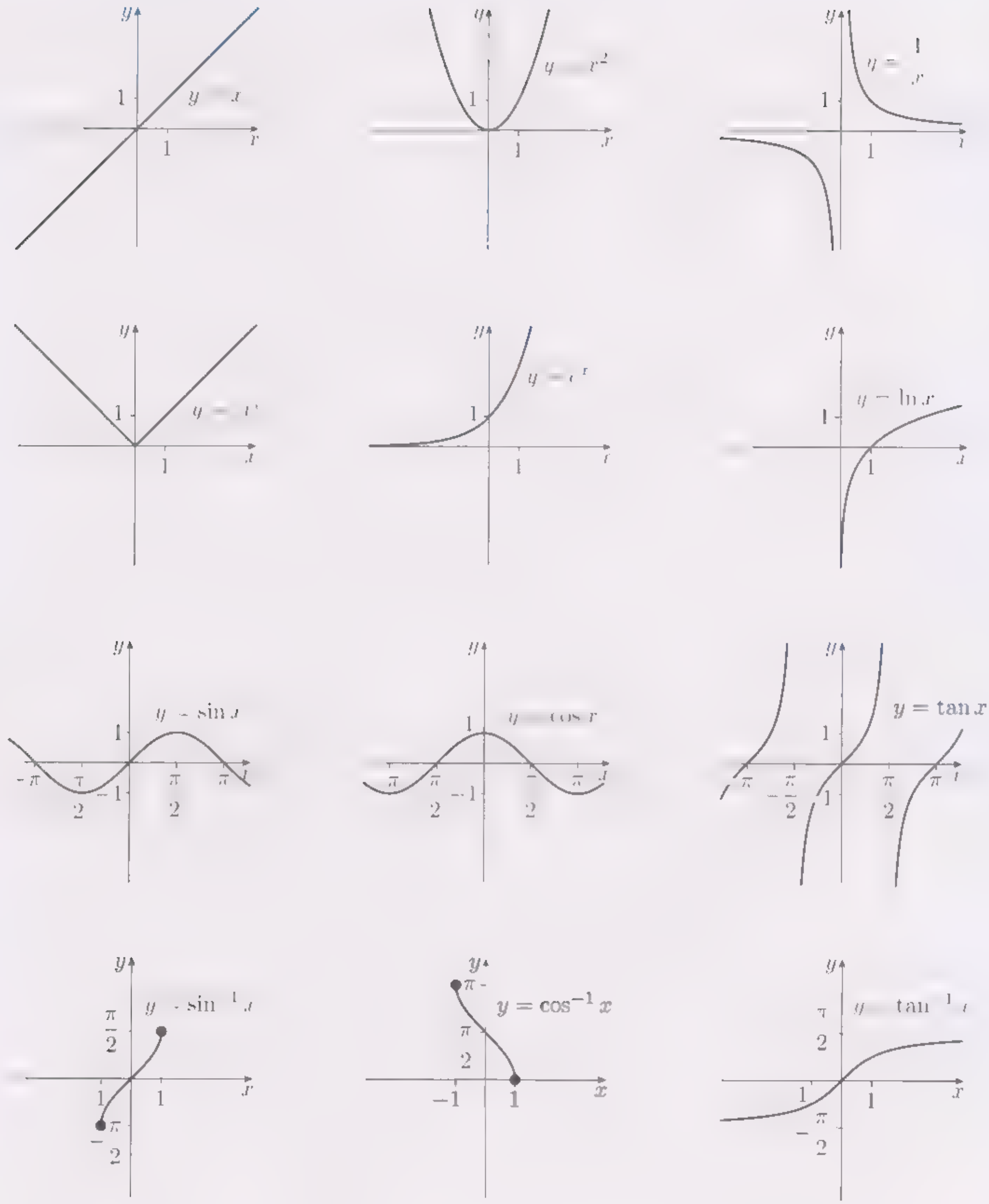
Sums of standard finite series

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

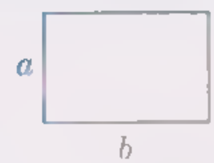
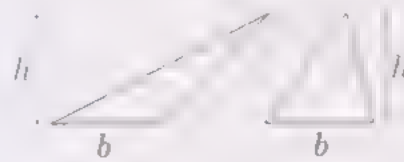
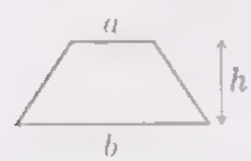
$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

Useful graphs



Geometry

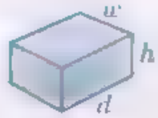
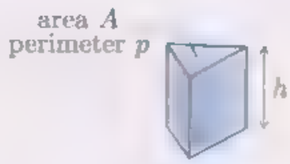
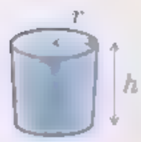
Areas

Shape		Area
Rectangle		ab
Parallelogram		bh
Triangle		$\frac{1}{2}bh$
		$\frac{1}{2}ab\sin\theta$
Trapezium		$\frac{1}{2}(a+b)h$

Circles

Shape		Formulas
Circle		circumference $= 2\pi r$ area $= \pi r^2$
Sector (θ in radians)		arc length $= r\theta$ area $= \frac{1}{2}r^2\theta$

Volumes and surface areas

Solid		Volume	Surface area
Cuboid		whd	$2wh + 2wd + 2hd$
Prism		Ah	$2A + hp$
Cylinder		$\pi r^2 h$	$2\pi r^2 + 2\pi rh$
Cone		$\frac{1}{3}\pi r^2 h$	$\pi r^2 + \pi rl$
Sphere		$\frac{4}{3}\pi r^3$	$4\pi r^2$

Opposite, corresponding and alternate angles

Where two lines cross,
opposite angles are equal.

Where a line crosses parallel lines,
corresponding angles are equal.

Where a line crosses parallel lines,
alternate angles are equal.

Greek alphabet

A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

SI units

The Système International d'Unités (SI units) is an internationally agreed set of units for measuring physical quantities.

The seven base units are as follows.

Unit	Symbol	Measurement of
metre	m	length
second	s	time
kilogram	kg	mass
kelvin	K	thermodynamic temperature
ampere	A	electric current
candela	cd	luminous intensity
mole	mol	amount of substance

Prefixes may be added to units. The commonly used prefixes are as follows.

Prefix	Symbol	Meaning	Example
nano	n	10^{-9}	nanogram, ng
micro	μ	10^{-6}	microsecond, μs
milli	m	10^{-3}	millisecond, ms
centi	c	10^{-2}	centimetre, cm
kilo	k	10^{3}	kilogram, kg
mega	M	10^{6}	megagram, Mg
giga	G	10^{9}	gigagram, Gg

There are also derived units, which are used for quantities whose measurement combines base units. Some of these are as follows.

Unit	Symbol	Meaning
area	m^2	metres squared or square metres
volume	m^3	metres cubed or cubic metres
speed	$m\,s^{-1}$	metres per second
acceleration	$m\,s^{-2}$	metres per second squared

A litre (l) is $0.001\,m^3$ (or $1000\,cm^3$). A metric tonne (t) is 1000 kg.

Notation

$\neq$	is not equal to
$\approx$	is approximately equal to
$<$	is less than
$\leq$	is less than or equal to
$>$	is greater than
$\geq$	is greater than or equal to
$\ldots$	ellipsis (dot, dot, dot), used when something has been left out
$\pm$	plus or minus
$\mp$	minus or plus
π	the ratio of the circumference of a circle to its diameter; $\pi = 3.141\,59\ldots$
e	the base for natural logarithms; $e = 2.718\,28\ldots$
∞	infinity
a^n	a to the power n
$\sqrt{a}$	the non-negative square root of the real number a, where $a \geq 0$
$\sqrt[n]{a}$	the non-negative nth root of the real number a, where $a \geq 0$
$\lvert x \rvert$	the modulus (magnitude, absolute value) of the real number x
$n!$	n factorial: $n! = 1 \times 2 \times \cdots \times n$; $0! = 1$
$\mathbb{N}$	the natural numbers: $1, 2, 3, \ldots$
$\mathbb{Z}$	the integers: $\ldots, -3, -2, -1, 0, 1, 2, 3\ldots$
$\mathbb{Q}$	the rational numbers: numbers of the form p/q, where p and q are integers, $q \neq 0$
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\in$	is a member of, 'is in'
$\notin$	is not a member of, 'is not in'
$\subseteq$	is a subset of
$\cap$	intersection (of sets)
$\cup$	union (of sets)
$\varnothing$	the empty set
$\circ$	the symbol for composition of functions; $(g \circ f)(x) = g(f(x))$
f^{-1}	the inverse function of the one-to-one function f
$\exp$	the exponential function
$\ln$	the natural logarithm function; $\log_e$
$\sin^2 \theta$	$(\sin\theta)^2$; a similar notation is used for other powers (but not -1) and for other trigonometric functions
$\sin^{-1} x$	the angle in the interval $[-\pi/2, \pi/2]$ whose sine is x
$\cos^{-1} x$	the angle in the interval $[0, \pi]$ whose cosine is x
$\tan^{-1} x$	the angle in the interval $(-\pi/2, \pi/2)$ whose tangent is x
$\angle ABC$	the angle formed by the line segments AB and BC
$\triangle ABC$	the triangle with vertices A, B and C

$\mathbf{i}$	the Cartesian unit vector in the direction of the x-axis
$\mathbf{j}$	the Cartesian unit vector in the direction of the y-axis
$\mathbf{k}$	the Cartesian unit vector in the direction of the z-axis
$\mathbf{0}$	the zero vector
$\lvert\mathbf{a}\rvert$	the magnitude (modulus) of the vector $\mathbf{a}$
$\overrightarrow{PQ}$	the displacement vector from P to Q
f'	the (first) derivative of the function f
f''	the second derivative of the function f
$f^{(n)}$	the nth derivative of the function f
$\dfrac{dy}{dx}$	the (first) derivative of y with respect to x
$\dfrac{d^2y}{dx^2}$	the second derivative of y with respect to x
$\dfrac{d^ny}{dx^n}$	the nth derivative of y with respect to x
$\dfrac{d}{dx}(y)$	a variant of $\dfrac{dy}{dx}$
$\displaystyle\int f(x)\,dx$	the indefinite integral of $f(x)$ with respect to x
$\displaystyle\int_a^b f(x)\,dx$	the definite integral of the function f from a to b
$\left[F(x)\right]_a^b$	$F(b) - F(a)$
$\mathbf{I}$	an identity matrix
$\mathbf{A}^{-1}$	the inverse of the invertible matrix $\mathbf{A}$
$\det \mathbf{A}$	the determinant of the square matrix $\mathbf{A}$
$\lvert\mathbf{A}\rvert$	the determinant of the square matrix $\mathbf{A}$
(x_n)	the infinite sequence $x_1, x_2, x_3, \ldots$
$(x_n)_{n=p}^{q}$	the finite sequence $x_p, x_{p+1}, x_{p+2}, \ldots, x_q$
$(x_n)_{n=p}^{\infty}$	the infinite sequence $x_p, x_{p+1}, x_{p+2}, \ldots$
$\displaystyle\sum_{n=p}^{q} x_n$	the finite sum $x_p + x_{p+1} + x_{p+2} + \cdots + x_q$
$\displaystyle\sum_{n=p}^{\infty} x_n$	the infinite sum $x_p + x_{p+1} + x_{p+2} + \cdots$
${}^nC_k$	a binomial coefficient; ${}^nC_k = \dfrac{n!}{k!\,(n-k)!}$
i	imaginary number whose square is -1
$\operatorname{Re} z$	the real part of the complex number z
$\operatorname{Im} z$	the imaginary part of the complex number z
$\bar{z}$	the complex conjugate of the complex number z
$\lvert z\rvert$	the modulus of the complex number z
$\operatorname{Arg} z$	the principal argument of the complex number z

Unit 1 Algebra

Numbers

Types of numbers

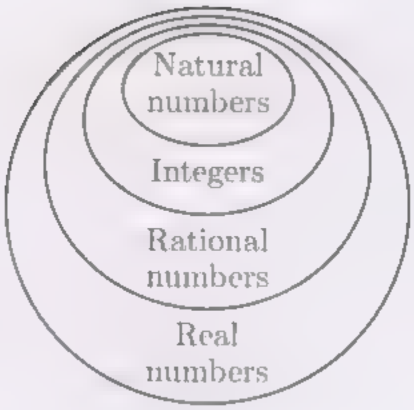
The **natural numbers** (or **positive integers**) are 1, 2, 3, ...

The **integers** are ..., −3, −2, −1, 0, 1, 2, 3, ...

The **rational numbers** are the numbers that can be written in the form integer/integer, or, equivalently, the decimal numbers that terminate or recur.

The **irrational numbers** are the numbers that cannot be written in the form integer/integer, or, equivalently, the decimal numbers with an infinite number of digits after the decimal point but with no block of digits that repeats indefinitely.

The **real numbers** are the rational numbers together with the irrational numbers. Each real number corresponds to a point on the number line.



A **prime number** (or **prime**) is an integer greater than 1 whose only positive factors are 1 and itself.

A **composite number** is an integer greater than 1 that is not a prime number.

A **square number** is an integer that can be written as the square of an integer.

The prime numbers under 100

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 97

The square numbers up to 15^2

0 1 4 9 16 25 36 49 64 81 100 121 144 169 196 225

The fundamental theorem of arithmetic

Every integer greater than 1 can be written as a product of prime factors in just one way (except that you can change the order of the factors).

Factors and multiples of integers

A **factor** of an integer is an integer that divides exactly into the first integer. A **multiple** of an integer is an integer into which the first integer divides exactly.

A **factor pair** of an integer is a pair of its factors that multiply together to give the integer. A **positive factor pair** of a positive integer is a pair of its positive factors that multiply together to give the integer.

The **prime factorisation** of an integer greater than 1 is any expression that shows it written as a product of prime factors.

Common factors and common multiples of integers

A **common multiple** of two or more integers is an integer that is a multiple of all of them. The **lowest** (or **least**) **common multiple** (**LCM**) of two or more integers is the smallest positive integer that is a multiple of all of them.

A **common factor** of two or more integers is an integer that is a factor of all of them. The **highest common factor** (**HCF**) (or **greatest common divisor (GCD)**) of two or more integers is the largest positive integer that is a factor of all of them.

To find the lowest common multiple or highest common factor of two or more integers greater than 1

- Find the prime factorisations of the numbers.
- To find the LCM, multiply together the highest power of each prime factor occurring in any of the numbers.
- To find the HCF, multiply together the lowest power of each prime factor common to all the numbers.

Powers and roots

The nth **power** of a number a, denoted by a^n, is obtained by multiplying together n copies of the number. Here a is the **base number** or **base**, and n is the **power**, **index** or **exponent**.

An nth **root** of a number is a number that when raised to the nth power gives the original number. A **square root** is a second root and a **cube root** is a third root. The non-negative square root of a non-negative number a is denoted by $\sqrt{a}$. The non-negative nth root of a non-negative number a is denoted by $\sqrt[n]{a}$.

The **reciprocal** of a number a is $1/a$, which can also be written as a^{-1}.

To express a number in **scientific notation**, write it in the form

(a number between 1 and 10, but not including 10)
× (an integer power of ten).

For the index laws, see page 6.

Surds

A **surd** is a numerical expression (such as $1 - 2\sqrt{5}$) that contains one or more irrational roots of numbers.

To manipulate surds, use the usual rules of algebra and the index laws.

To **rationalise** the denominator of a surd that has an irrational denominator, multiply the numerator and denominator by a suitable surd. If the denominator is is a sum of two terms, either or both of which is a rational number multiplied by an irrational square root, then a suitable surd is the expression obtained by changing the sign of one of the two terms in the sum. This expression is called a **conjugate** of the expression in the denominator. (For example, a conjugate of $\sqrt{2} - 3\sqrt{5}$ is $\sqrt{2} + 3\sqrt{5}$.)

Expressions

An **expression** is an arrangement of letters, numbers and/or mathematical symbols, which is such that if values are substituted for any letters present, then you can work out the value of the arrangement.

An **algebraic expression** includes letters. Each such letter may be:

- a **variable**, representing any number, or any number of a particular kind
- an **unknown**, representing a particular number that you do not know, but usually you want to discover
- a **constant**, representing a particular number whose value is fixed.

If an expression is a list of quantities that are added, then these quantities are called the **terms** of the expression.

If a term consists of a number multiplied by powers of variables, then the number is called the **coefficient** of the term.

An **algebraic fraction** is an algebraic expression written in the form of a fraction. The **numerator** and **denominator** of the fraction are its top and bottom, respectively.

Factors and multiples of algebraic expressions

Roughly speaking, if an algebraic expression can be written in the form

something $\times$ something,

then both 'somethings' are **factors** of the expression, and the expression is a **multiple** of both 'somethings'.

A **common factor** of two or more algebraic expressions is an expression that is a factor of all of them.

A **common multiple** of two or more algebraic expressions is an expression that is a multiple of all of them.

A **highest common factor** of two or more algebraic expressions is a common factor that is a multiple of all other common factors.

A **lowest common multiple** of two or more algebraic expressions is a common multiple that is a factor of all other common multiples.

Factorising an expression means writing it as the product of two or more expressions, neither of which is 1 (and, usually, neither of which is -1).

Difference of two squares

$(A+B)(A-B) = A^2 - B^2$

Squaring brackets

$(A+B)^2 = A^2 + 2AB + B^2$

$(A-B)^2 = A^2 - 2AB + B^2$

Equations

An **equation** consists of two expressions, with an equals sign between them.

The **solutions** of an equation are the values of its variables that make the equation true. These values **satisfy** the equation.

An **identity** is an equation that is satisfied by all possible values of its variables.

A **linear equation** is an equation in which, after you have expanded any brackets and cleared any fractions, each term is either a constant term or a constant value times a variable.

If an equation contains more than one variable, and one side of the equation is just a single variable that does not appear at all on the other side, then that variable is called the **subject** of the equation.

To rearrange an equation

Carry out any of the following operations on an equation to obtain an equivalent equation.

- Rearrange the expressions on one or both sides.
- Swap the sides.
- Do any of the following things to both sides:
  - add or subtract something
  - multiply or divide by something (provided that it is non-zero)
  - raise to a power (provided that the power is non-zero, and that the expressions on each side of the equation can take only non-negative values).

To make a variable the subject of an equation

(This works for some equations but not all.)

Use the rules for rearranging equations to obtain successive equivalent equations. Aim to obtain an equation in which the required subject is alone on one side. To achieve this, do the following, in order.

1. Clear any fractions and multiply out any brackets. To clear fractions, multiply through by a suitable expression.
2. Add or subtract terms on both sides to get all the terms containing the required subject on one side, and all the other terms on the other side. Collect like terms.
3. If more than one term contains the required subject, then take it out as a common factor.
4. Divide both sides by the expression that multiplies the required subject.

Unit 2 Graphs and equations

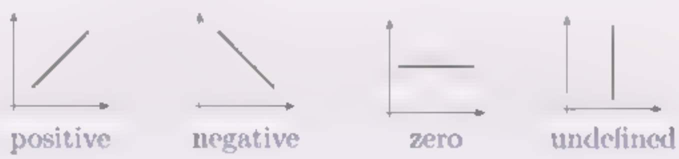
Mathematical modelling

A **mathematical model** is a collection of assumptions and mathematical statements that is intended to describe how some phenomenon in the real world behaves, and to enable predictions to be made about its behaviour.

A **linear mathematical model** is a mathematical model based on a relationship between two variables that is represented by a straight line.

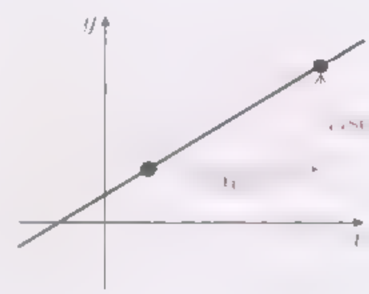
Straight-line graphs

The **gradient** of a straight line is the number of units moved up for every one unit moved to the right.



The gradient of the line through the points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is given by

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$



The x- and y-**intercepts** of a straight line are the values of x and y where the line crosses the x-axis and the y-axis, respectively.

Equations of lines

The straight line with gradient m and y-intercept c has equation

$$y = mx + c.$$

The horizontal line with y-intercept c has equation $y = c$.

The vertical line with x-intercept d has equation $x = d$.

The straight line with gradient m that passes through the point (x_1, y_1) has equation

$$y - y_1 = m(x - x_1).$$

Parallel and perpendicular lines

Two straight lines are **parallel** if they never cross, even when extended infinitely far in each direction. The gradients of any two parallel lines are equal (or both lines are vertical).

Two straight lines are **perpendicular** if they are at right angles to each other. The gradients of any two perpendicular lines (not parallel to the axes) have product -1.

Simultaneous linear equations

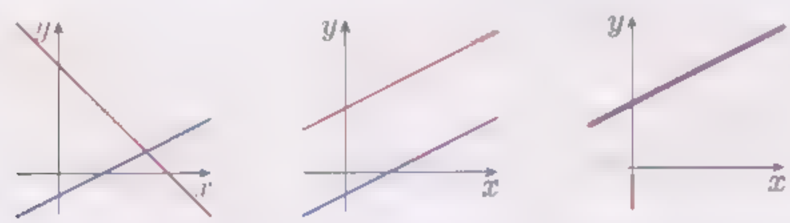
A pair of **simultaneous linear equations** consists of two equations of the form

$$ax + by = e$$
$$bx + cy = f$$

that apply simultaneously. Here x and y are unknowns, and a, b, c, d, e and f are constants.

A **solution** of a pair of simultaneous equations is a pair of values of the unknowns x and y that satisfy both equations. These values are the coordinates of the point of intersection of the lines represented by the equations.

A pair of simultaneous linear equations may have one solution, no solutions, or infinitely many solutions.



One solution — No solutions — Infinitely many solutions

To solve a pair of simultaneous equations: substitution method

1. Rearrange one of the equations, if necessary, to obtain a formula for one unknown in terms of the other.
2. Use this formula to substitute for this unknown in the other equation.
3. You now have an equation in one unknown. Solve it to find the value of that unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: confirm that the two values satisfy the original equations.)

To solve a pair of simultaneous equations: elimination method

1. Multiply one or both of the equations by suitable numbers, if necessary, to obtain two equations that can be added or subtracted to eliminate one of the unknowns.
2. Add or subtract the equations to eliminate this unknown.
3. You now have an equation in one unknown. Solve it to find the value of that unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: confirm that the two values satisfy the original equations.)

Quadratic expressions

A **quadratic expression**, or **quadratic**, is an expression of the form $ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$.

To factorise a quadratic of the form $x^2 + bx + c$

1. Start by writing $x^2 + bx + c = (x \quad)(x \quad)$.
2. Find the factor pairs of c (including both positive and negative ones).
3. Choose the factor pair with sum b, if there is such a pair.
4. Write your factor pair p, q in position: $x^2 + bx + c = (x + p)(x + q)$.

To factorise a quadratic of the form $ax^2 + bx + c$

1. Take out any numerical common factors. If the coefficient of x^2 is negative, also take out the factor -1. Then apply the steps below to the quadratic inside the brackets.
2. Find the positive factor pairs of a, the coefficient of x^2. For each such factor pair d, e write down a framework $(dx \quad)(ex \quad)$.
3. Find all the factor pairs of c, the constant term (including both positive and negative ones).
4. For each framework and each factor pair of c, write the factor pair in the gaps in the framework in both possible ways.
5. For each of the resulting cases, calculate the term in x that you obtain when you multiply out the brackets.
6. Identify the case where this term is bx, if there is such a case. This is the required factorisation.

To complete the square in a quadratic of the form $x^2 + bx$

1. Write down $(x \quad)^2$, filling the gap with the number that is half of b, the coefficient of x (including its $+$ or $-$ sign).
2. Subtract the square of the number that you wrote in the gap.

To complete the square in a quadratic of the form $x^2 + bx + c$

1. Use the strategy above to complete the square in the subexpression $x^2 + bx$.
2. Collect the constant terms.

To complete the square in a quadratic of the form $ax^2 + bx + c$, where $a \neq 1$

1. Rewrite the quadratic with the coefficient a taken out of the subexpression $ax^2 + bx$ as a factor. This generates a pair of brackets.
2. Use the earlier strategy to complete the square in the simple quadratic inside the brackets. This generates a second pair of brackets, inside the first pair.
3. Multiply out the outer brackets.
4. Collect the constant terms.

Quadratic equations

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$. It has at most two solutions.

To simplify a quadratic equation

- If necessary, rearrange the equation so that all the non-zero terms are on the same side.
- If the coefficient of x^2 is negative, then multiply the equation through by -1 to make this coefficient positive.
- If the coefficients have a common factor, then divide the equation through by this factor.
- If any of the coefficients are fractions, then multiply the equation through by a suitable number to clear them.

To solve a quadratic equation $ax^2 + bx + c = 0$

Simplify it, then use one of the following three methods.

Factorisation

1. Factorise the quadratic expression.
2. Use the fact that if the product of two numbers is zero then at least one of the numbers must be zero.
3. Solve the resulting two linear equations.

Completing the square

1. Complete the square in the quadratic expression.
2. Rearrange the equation to obtain the square alone on the left-hand side and a constant on the right-hand side.
3. Take square roots of both sides (remembering that a positive number has both a positive and a negative square root).
4. Solve the resulting two linear equations.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The number of real solutions of a quadratic equation

The **discriminant** of the quadratic expression $ax^2 + bx + c$ is the value $b^2 - 4ac$.

The quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) has:

- two real solutions if $b^2 - 4ac > 0$ (the discriminant is positive)
- one real solution if $b^2 - 4ac = 0$ (the discriminant is zero)
- no real solutions if $b^2 - 4ac < 0$ (the discriminant is negative).

A **repeated solution** of a quadratic equation is a solution that is the only solution of the equation.

Quadratic graphs

The shape of the graph of an equation of the form $y = ax^2 + bx + c$, where a, b and c are constants with $a \neq 0$, is called a **parabola**.

a positive **u-shaped** parabola

a negative **n-shaped** parabola

Properties of the graph of $y = ax^2 + bx + c$, where $a \neq 0$

- The graph is a parabola with a vertical axis of symmetry.
- If a is positive it is **u-shaped**, if a is negative it is **n-shaped**.
- It has two, one or no x-intercepts .
- It has one y-intercept.

u-shaped parabolas with two, one and no x-intercepts

To sketch the graph of $y = ax^2 + bx + c$, where $a \neq 0$

1. Find whether the parabola is u-shaped or n-shaped.
2. Find its intercepts, axis of symmetry and vertex.
3. Plot the features found, and hence sketch the parabola.
4. Label the parabola with its equation, its intercepts and the coordinates of its vertex.

Sketching versus plotting

A **sketch** of a graph is a diagram that gives an impression of its shape, with key points marked and positioned approximately correctly relative to a pair of coordinate axes.

A **plot** of a graph is a more accurate diagram obtained by precisely plotting a reasonably large number of points on the graph.

Displacement and velocity along a straight line

For an object moving along a straight line,

- its **displacement** from a reference point is its distance from the reference point, with a plus or minus sign to indicate the direction
- its **velocity** is its speed, with a plus or minus sign to indicate the direction.

Unit 3 Functions

Sets

A **set** is a collection of objects.

The notation $x \in A$ means that the object x is in the set A.

If A and B are any two sets, then

- their **intersection** $A \cap B$ is the set whose members are all the elements that belong to both A and B
- their **union** $A \cup B$ is the set whose members are all the elements that belong to either A or B (or both).

These definitions extend to intersections and unions of more than two sets.

A **subset** of a set A is a set whose elements all belong to A.

The **empty set**, denoted by $\varnothing$, is the set that contains no elements.

The set containing the single element a can be denoted by $\{a\}$. Similarly, the set containing the elements a and b can be denoted by $\{a, b\}$, and so on.

Intervals

An **interval** is a set of real numbers that corresponds to a part of the number line that you can draw 'without lifting your pen from the paper'. A number that lies at an end of an interval is called an **endpoint**.

A **closed** interval includes all of its endpoints.

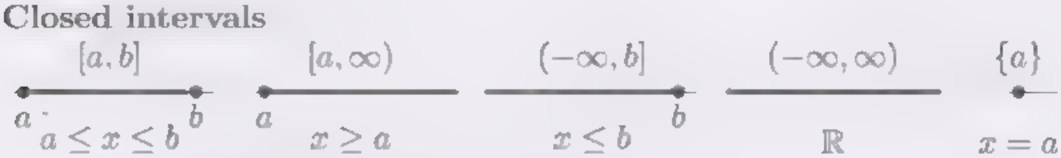
An **open** interval does not include any of its endpoints.

A **half-open** (or **half-closed**) interval includes one endpoint and excludes another.

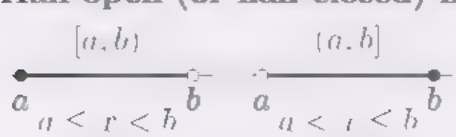
The interval $\mathbb{R}$ has no endpoints, so it is both open and closed.

Open intervals

Closed intervals



Half-open (or half-closed) intervals



Functions

A **function** consists of:

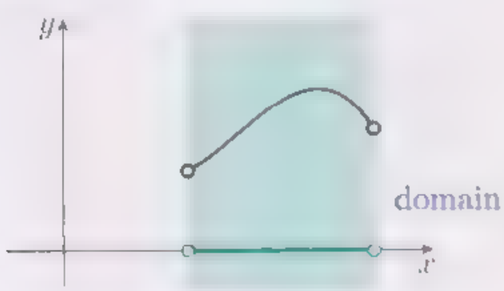
- a set of allowed input values, called the **domain** of the function
- a set of values in which every output value lies, called the **codomain** of the function
- a process, called the **rule** of the function, for converting each input value into exactly one output value.

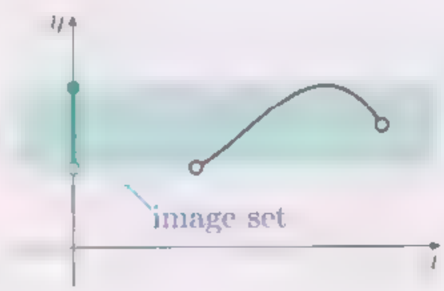
If f is a function and x is a value in its domain, then the **image of x under f**, or the **value of f at x**, denoted by $f(x)$, is the output value corresponding to the input value x.

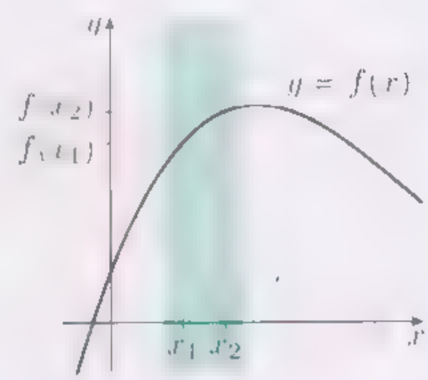
The **image set** of a function is the set consisting of all the values in its codomain that occur as output values.

A **real function** is a function whose domain and codomain are both subsets of $\mathbb{R}$. In MST124 the word 'function' is assumed to mean 'real function', and it is assumed that the codomain of every function is $\mathbb{R}$.

The **graph** of a function f consists of the points with coordinates $(x, f(x))$, where x is a value in the domain of f.



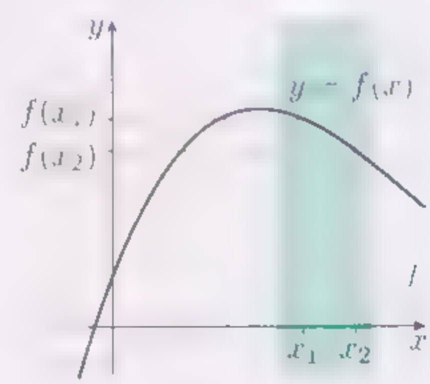




A function f increasing on an interval I

When a function is specified using an equation that expresses one variable in terms of another variable, the input variable is called the **independent variable** and the output variable is called the **dependent variable**.

A **piecewise-defined function** is a function whose rule is specified by using different formulas for different parts of its domain.



A function f decreasing on an interval I

Domain convention

When a function is specified by just a rule, it is understood that the domain of the function is the largest possible set of real numbers for which the rule is applicable.

Functions increasing or decreasing on an interval

A function f is **increasing on the interval I** if for all values x_1 and x_2 in I such that $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing on the interval I** if for all values x_1 and x_2 in I such that $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

Composite functions

The **composite function** $g \circ f$ of the functions f and g is the function whose rule is

$$(g \circ f)(x) = g(f(x)),$$

and whose domain consists of all the values x in the domain of f such that $f(x)$ is in the domain of g.

Inverse functions

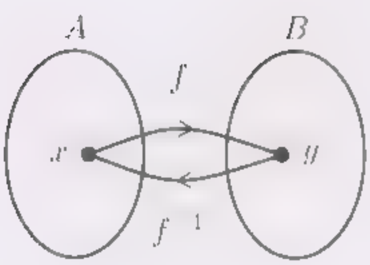
One-to-one functions

A function f is **one-to-one** (or **invertible**) if for all numbers x_1 and x_2 in its domain such that $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$. (That is, no two different input values give the same output value.)

Only one-to-one functions have inverse functions.

If a function is either increasing on its whole domain or decreasing on its whole domain, then it is one-to-one.

Inverse functions



If f is a one-to-one function with domain A and image set B, then the **inverse function**, or **inverse**, of f, denoted by f^{-1}, is the function with domain B whose rule is given by

$$f^{-1}(y) = x, \quad \text{where} \quad f(x) = y.$$

The image set of f^{-1} is A.

If a function f has inverse f^{-1}, then the function f^{-1} has inverse f.

For any pair of inverse functions f and f^{-1},

$$(f^{-1} \circ f)(x) = x, \quad \text{for every value } x \text{ in the domain of } f \text{, and}$$
$$(f \circ f^{-1})(x) = x, \quad \text{for every value } x \text{ in the domain of } f^{-1}.$$

To find the rule of the inverse function of a one-to-one function

1. Write $y = f(x)$ and rearrange this equation to express x in terms of y.
2. Use the resulting equation $x = f^{-1}(y)$ to write down the rule of f^{-1}. (Usually, change the input variable from y to x.)

Graphs of inverse functions

The graphs of every pair of inverse functions are the reflections of each other in the line $y = x$ (when the coordinate axes have equal scales).

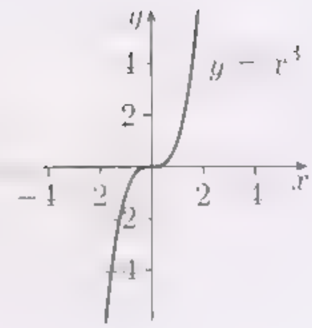
Functions that are not one-to-one

A **restriction** of a function is a function obtained by keeping the rule the same, but removing some values from the domain.

For a function that is not one-to-one and so has no inverse function, we sometimes use the inverse function of a one-to-one restriction of the function.

Some standard functions

Polynomial functions



A **polynomial expression** in x is a sum of finitely many terms, each of the form ax^n where a is a number and n is a non-negative integer.

A **polynomial function** is a function whose rule is of the form '$f(x) =$ a polynomial expression in x'.

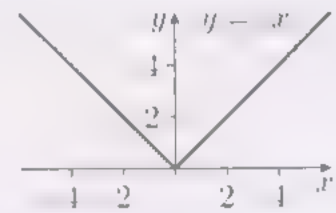
The **degree** of a polynomial expression or function is the highest power of the variable (usually x) in the expression or function.

A **linear function** is a polynomial function of degree at most 1.

A **quadratic**, **cubic**, **quartic** or **quintic** function is a polynomial function of degree 2, 3, 4 or 5, respectively.

The terms **linear**, **quadratic**, **cubic**, **quartic** and **quintic** are used for polynomial expressions in the same way.

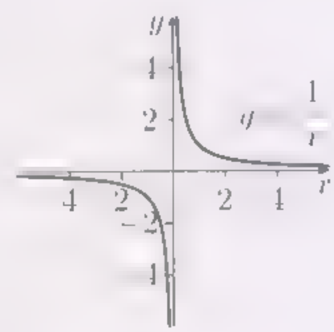
Modulus function



The **modulus**, **magnitude** or **absolute value** $|x|$ of a real number x is its 'distance from zero', or its 'value without its sign'. For example, $|3| = 3$ and $|-3| = 3$.

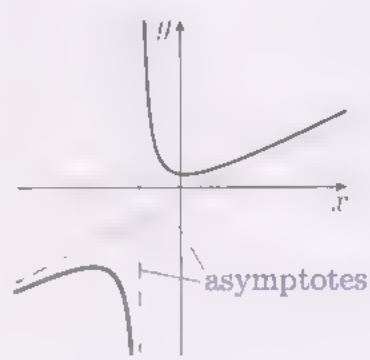
The **modulus function** is the function $f(x) = |x|$.

Reciprocal function



The **reciprocal function** is the function $f(x) = 1/x$. Its domain contains all real numbers except 0.

Rational functions



A **rational function** is a function whose rule is of the form $f(x) = p(x)/q(x)$, where p and q are polynomial functions.

The reciprocal function and every polynomial function are rational functions.

An **asymptote** is a line that a curve approaches arbitrarily closely as the distance along the curve away from the origin increases.

Many rational functions have asymptotes, which can be horizontal, vertical or slant.

Exponential and logarithmic functions

For details of these functions, see page 30.

Trigonometric functions

For details of these functions, see page 35.

Translating, reflecting and scaling graphs

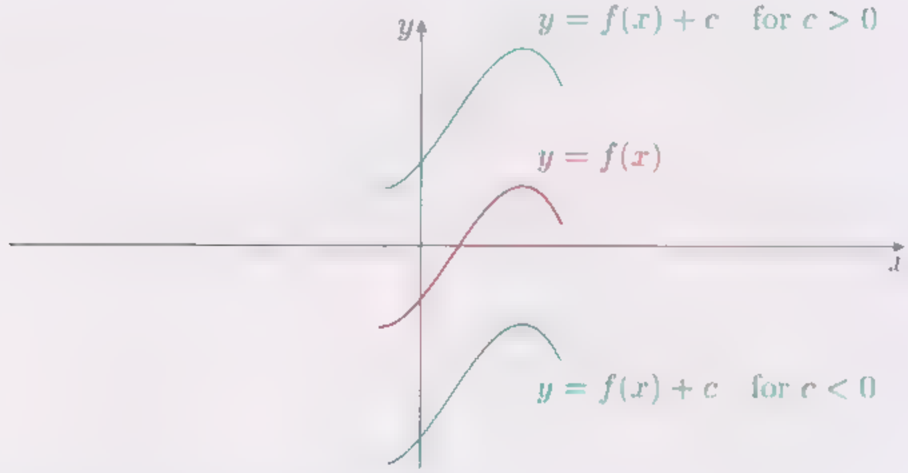
On this page and the next, f is a function and c is a constant.

The scalings and translations on these pages can be applied in any order with the same result, except that the order of a vertical translation and a vertical scaling relative to each other affects the result, and similarly for a horizontal translation and a horizontal scaling.

Translating graphs vertically

To obtain the graph of $y = f(x) + c$,

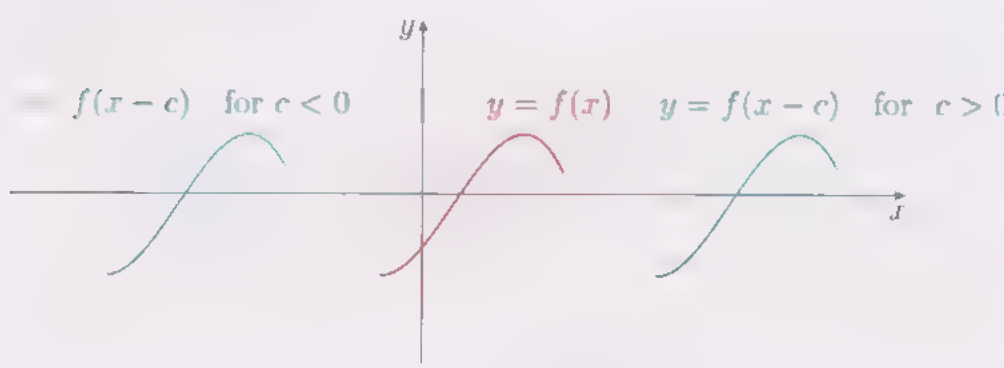
> **translate** the graph of $y = f(x)$ up by c units
> (the translation is down if c is negative).



Translating graphs horizontally

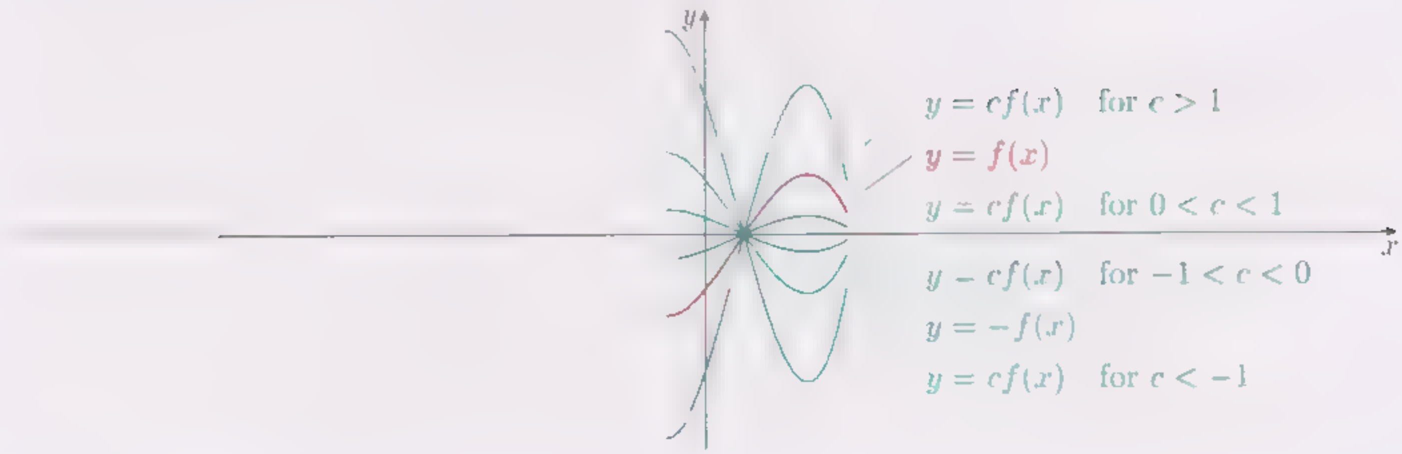
To obtain the graph of $y = f(x - c)$,

> **translate** the graph of $y = f(x)$ to the right by c units
> (the translation is to the left if c is negative).



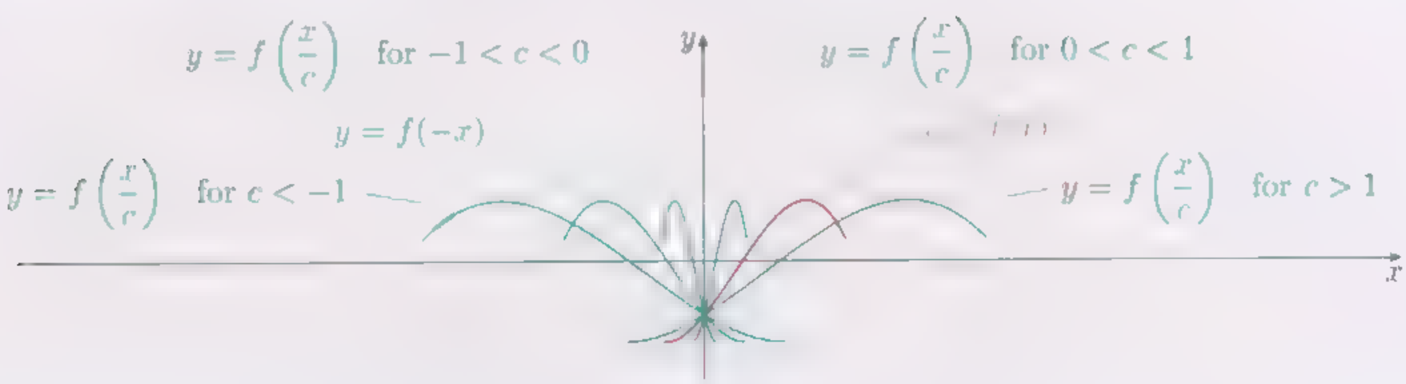
Scaling graphs vertically

To obtain the graph of $y = cf(x)$,
scale the graph of $y = f(x)$ vertically by a factor of c.



Scaling graphs horizontally

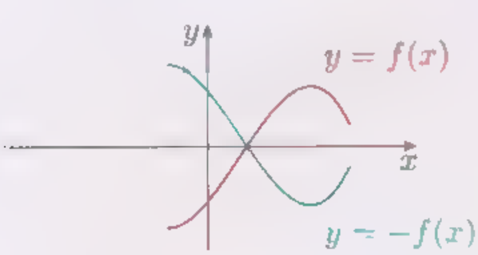
To obtain the graph of $y = f\left(\dfrac{x}{c}\right)$,
scale the graph of $y = f(x)$ horizontally by a factor of c.

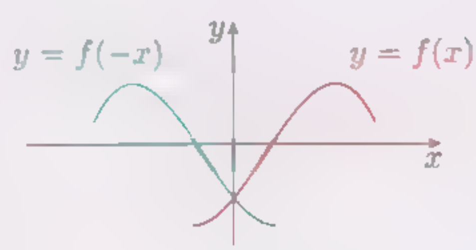


Reflecting graphs in the coordinate axes

To obtain the graph of

- $y = -f(x)$, **reflect** the graph of $y = f(x)$ in the x-axis
- $y = f(-x)$, **reflect** the graph of $y = f(x)$ in the y-axis.



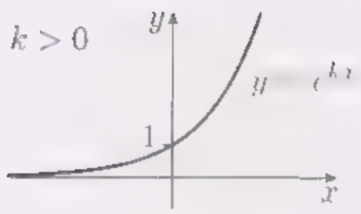


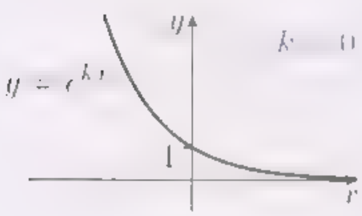
Exponential functions

An **exponential function** is a function whose rule is of the form

$$f(x) = b^x,$$

where b is a positive constant, not equal to 1. The number b is the **base number** or **base** of the exponential function.





Equivalently, an **exponential function** is a function whose rule is of the form

$$f(x) = e^{kx},$$

where k is a non-zero constant (and e is the special constant $2.718\ldots$).

The exponential function

***The* exponential function** is the function $f(x) = e^x$.
It has the special property that its gradient is exactly 1 at the point $(0, 1)$.
The expression e^x can be written as $\exp x$ or $\exp(x)$.

Logarithms

The **logarithm to base b** of a number x, denoted by $\log_b x$, is the power to which the base b must be raised to give the number x. So the equations

$$y = \log_b x \quad \text{and} \quad x = b^y$$

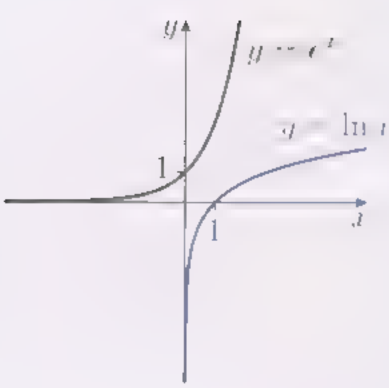
are equivalent. The base b must be positive and not equal to 1.

Only positive numbers have logarithms, but logarithms themselves can be any number.

A **logarithmic function** is a function whose rule is of the form

$$f(x) = \log_b x,$$

where b is a positive constant, not equal to 1. The number b is the **base** or **base number** of the logarithmic function. The functions $f(x) = \log_b x$ and $g(x) = b^x$ are inverses of each other.



Natural logarithms

The **natural logarithm** of a number x, denoted by $\ln x$, is its logarithm to base e. That is, it is the power to which e must be raised to give the number x. So the equations

$$y = \ln x \quad \text{and} \quad x = e^y$$

are equivalent.

The **natural logarithm function** is the function $f(x) = \ln x$.
The functions $f(x) = \ln x$ and $g(x) = e^x$ are inverses of each other.

For the logarithm laws, see page 6.

Exponential equations

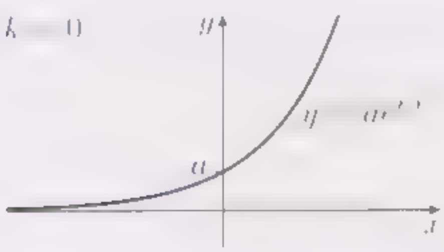
An **exponential equation** is an equation in which an unknown is in an exponent. To solve an exponential equation, take logarithms of both sides.

Exponential growth and decay

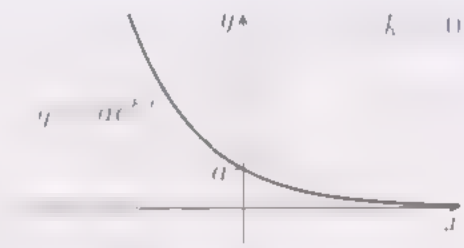
A quantity **changes exponentially** if its change can be modelled by a function whose rule is of the form $f(x) = ae^{kx}$, where a and k are non-zero constants.

If a and k are both positive, then the quantity **grows exponentially**, the function is an **exponential growth function**, and the graph is an **exponential growth curve**.

If a is positive but k is negative, then the quantity **decays exponentially**, the function is an **exponential decay function**, and the graph is an **exponential decay curve**.



Exponential growth curve



Exponential decay curve

Growth and decay factors

If $f(x) = ae^{kx}$, then whenever x is increased by p units, the value of $f(x)$ is multiplied by e^{kp}.

If $p > 0$, this factor e^{kp} is called the **growth factor** of f for the period p or the **decay factor** of f for the period p, according to whether f is an exponential growth or decay function. Growth factors are greater than 1, and decay factors are between 0 and 1, exclusive.

Doubling and halving periods

The **doubling period** of the exponential growth function f is the value of p such that whenever x is increased by p the value of $f(x)$ doubles.

The **halving period** of the exponential decay function f is the value p such that whenever x is increased by p the value of $f(x)$ halves.

To find a doubling or halving period

If $f(x) = ae^{kx}$ is an exponential growth function (so $k > 0$), then the doubling period of f is the solution p of the equation $e^{kp} = 2$; that is, $p = (\ln 2)/k$.

Similarly, if $f(x) = ae^{kx}$ is an exponential decay function (so $k < 0$), then the halving period of f is the solution p of the equation $e^{kp} = \frac{1}{2}$; that is, $p = (\ln \frac{1}{2})/k = -(\ln 2)/k$.

Inequalities

An **inequality** consists of two expressions, with one of the four inequality signs between them ($<$, $\leq$, $>$ or $\geq$).

The **solutions** of an inequality are the values of its variables that make the inequality true. These values **satisfy** the inequality.

The **solution set** of an inequality is the set formed by its solutions.

To rearrange an inequality

Carry out any of the following operations on an inequality to obtain an equivalent inequality.

- Rearrange the expressions on one or both sides.
- Swap the sides, *provided you reverse the inequality sign.*
- Do any of the following things to both sides:
  - add or subtract something
  - multiply or divide by something that is positive
  - multiply or divide by something that is negative, *provided you reverse the inequality sign.*

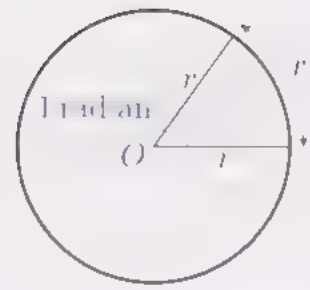
To solve an inequality using a table of signs

1. Rearrange the inequality to obtain an algebraic expression on one side and only 0 on the other side.
2. If the expression contains any algebraic fractions, then rearrange the whole expression into a single algebraic fraction.
3. Factorise the expression as far as possible.
4. Construct a table of signs:
   (a) In the column headings, write (in increasing order) the values of the variable for which the factors of the expression are equal to zero, and also the largest open intervals to the left and right of, and between, these values.
   (b) In the row headings, write each of the factors of the expression, and then the whole expression.
   (c) For each factor, write 0, $+$ or $-$ in each cell in its row, to indicate whether the factor is zero, positive or negative for the indicated values of the variable.
   (d) Use the signs of the factors to find the signs of the whole expression, and enter these in the bottom row. (Where the expression is undefined, enter the symbol $*$ to indicate this.)
5. Use the entries in the bottom row to solve the inequality.

Unit 4 Trigonometry

Angles

Radians

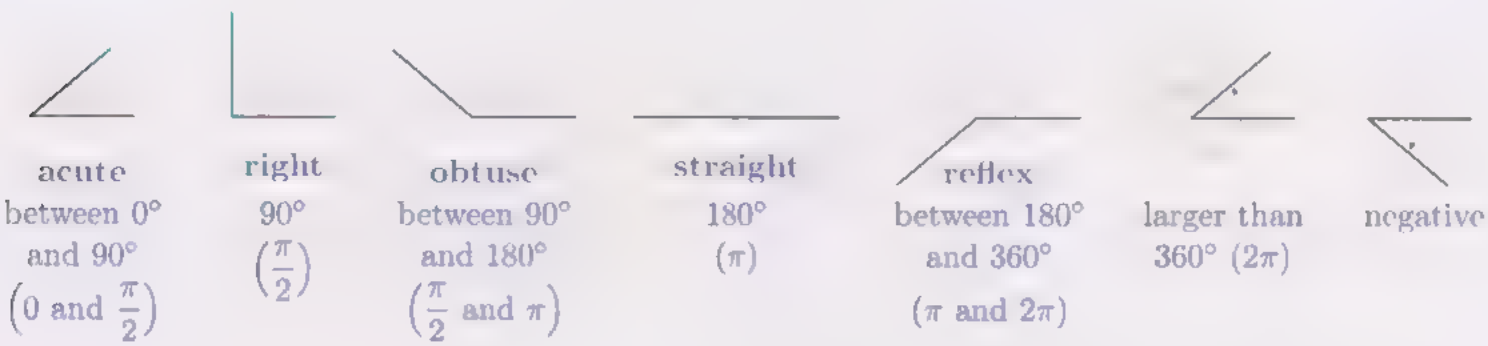


One **radian** is the angle subtended at the centre of a circle by an arc that has the same length as the radius.

So 2π radians $= 360°$.

If the size of an angle is stated with no units, then the angle is measured in radians.

Types of angle



Right-angled triangles

A **right-angled triangle** is a triangle that contains a right angle. Each of the other two angles in the triangle is an acute angle.

The **hypotenuse** of a right-angled triangle is the side opposite the right angle. It is always the longest side.

Pythagoras' theorem

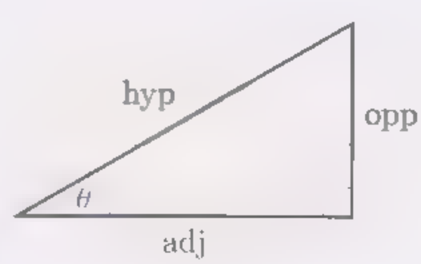
For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Trigonometric ratios for right-angled triangles

The **sine**, **cosine** and **tangent** of the acute angle θ are given by

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

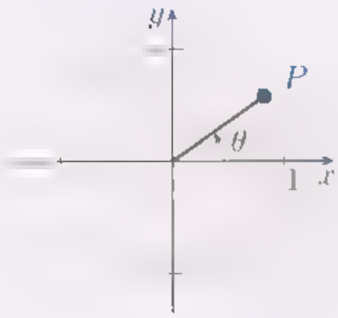
Mnemonic: SOH CAH TOA.



Here hyp, opp and adj denote the hypotenuse, the side opposite θ and the side adjacent to θ, respectively, in a right-angled triangle with an acute angle θ.

For trigonometric ratios for special angles, see page 4.

Trigonometric ratios for angles of any size



The **unit circle** is the circle of radius 1 centred at the origin.

With every angle θ we associate a point P on the unit circle. The point P is obtained by a rotation around the origin through the angle θ, starting from the point on the x-axis with x-coordinate 1. If θ is positive, then the rotation is anticlockwise; if θ is negative, then the rotation is clockwise.

Suppose that θ is any angle and (x, y) are the coordinates of its associated point P on the unit circle. Then

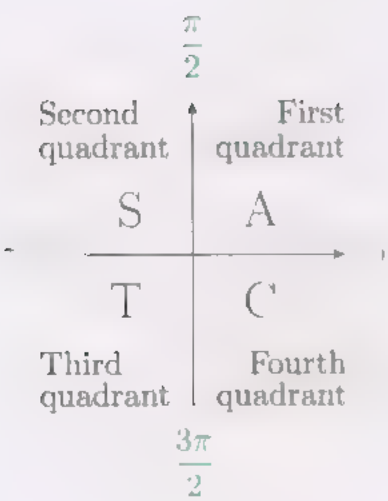
$$\sin\theta = y, \quad \cos\theta = x,$$

and, provided that $x \neq 0$,

$$\tan\theta = \frac{y}{x}.$$

(If $x = 0$, then $\tan\theta$ is undefined.)

The ASTC diagram

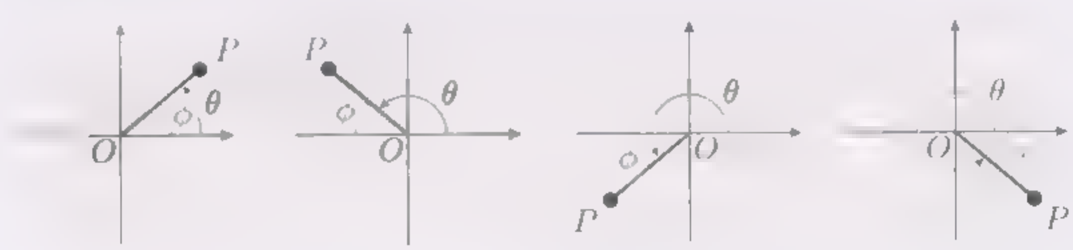


A **quadrant** is one of the four regions separated off by the x- and y-axes.

The **ASTC diagram** indicates which of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive when the point P associated with the angle θ lies in each quadrant. (A stands for all, S stands for sin, T stands for tan, C stands for cos.)

Relationships between trigonometric ratios of angles in different quadrants

Suppose that θ is an angle whose associated point P does not lie on either the x- or y-axis, and ϕ is the acute angle between OP and the x-axis, as in the examples below.



Then

$$\sin\theta = \pm\sin\phi \qquad \cos\theta = \pm\cos\phi \qquad \tan\theta = \pm\tan\phi.$$

The ASTC diagram tells you which sign applies in each case.

(The values $\sin\phi$, $\cos\phi$ and $\tan\phi$ are all positive, because ϕ is acute.)

Two basic trigonometric identities

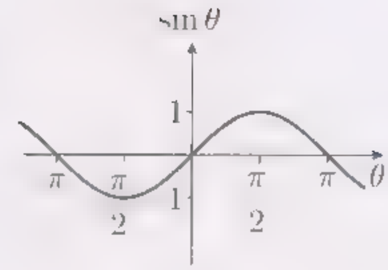
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

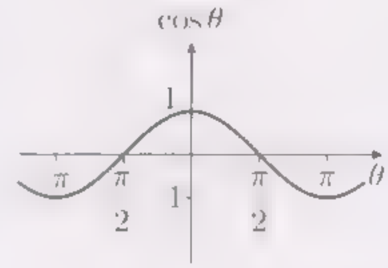
For more trigonometric identities, see page 5.

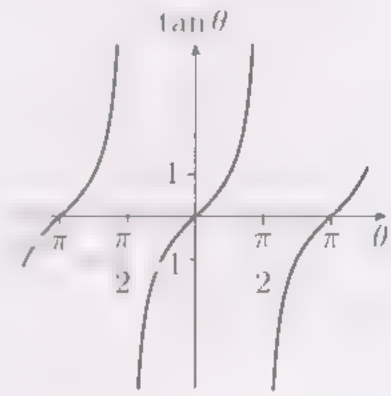
Trigonometric functions

Sine, cosine and tangent functions

The graphs of the sine and cosine functions are **periodic** with **period** 2π (they repeat their shape after every interval of 2π). The graph of the tangent function is periodic with period π.







Inverse sine, cosine and tangent functions

The **inverse sine function** $\sin^{-1}$ has domain $[-1,1]$ and rule

$$\sin^{-1} x = y,$$

where y is the number in the interval $[-\pi/2, \pi/2]$ such that $\sin y = x$.

The **inverse cosine function** $\cos^{-1}$ has domain $[-1,1]$ and rule

$$\cos^{-1} x = y,$$

where y is the number in the interval $[0, \pi]$ such that $\cos y = x$.

The **inverse tangent function** $\tan^{-1}$ has domain $\mathbb{R}$ and rule

$$\tan^{-1} x = y,$$

where y is the number in the interval $(-\pi/2, \pi/2)$ such that $\tan y = x$.

The inverse sine, inverse cosine and inverse tangent functions are also called the **arcsine**, **arccosine** and **arctangent** functions.

For the graphs of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ see page 9.

Identities for inverse trigonometric functions

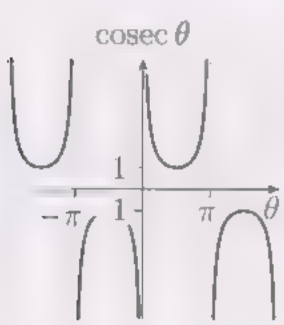
$$\sin^{-1}(-x) = -\sin^{-1} x$$
$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$
$$\tan^{-1}(-x) = -\tan^{-1} x$$

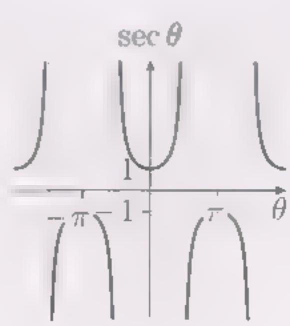
Cosecant, secant and cotangent functions

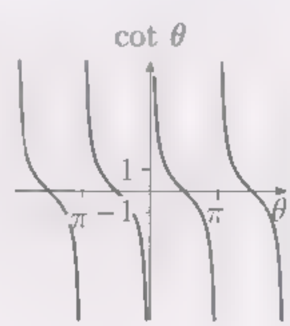
The **cosecant**, **secant** and **cotangent** functions are given by

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

(So $\cot\theta = 1/(\tan\theta)$, if θ is not an integer multiple of $\pi/2$.)







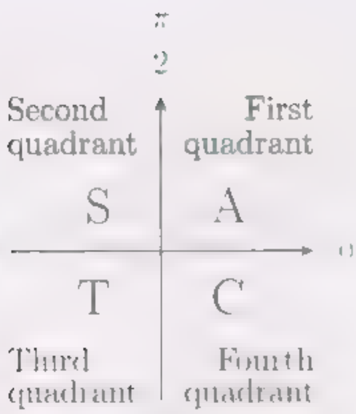
Trigonometric equations

A **trigonometric equation** is an equation that contains a trigonometric function of an unknown.

A trigonometric equation of the form

$$\sin\theta = c, \quad \cos\theta = c \quad \text{or} \quad \tan\theta = c,$$

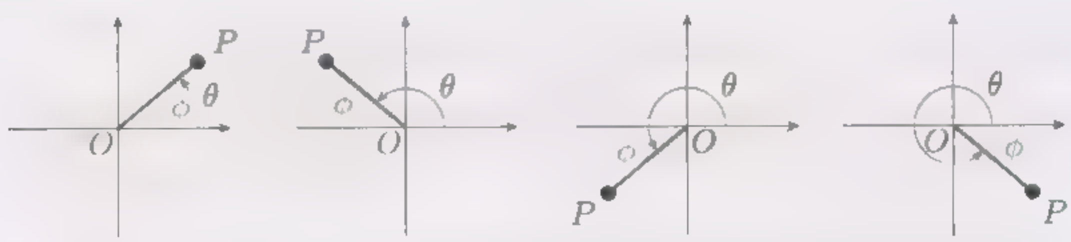
where c is a constant and θ is the unknown, has infinitely many solutions.



To solve an equation of the form $\sin\theta = c$, $\cos\theta = c$ or $\tan\theta = c$ by using the ASTC diagram

(This method does not apply if $c = 0$, or if the equation is $\sin\theta = c$ or $\cos\theta = c$ where $c = \pm 1$.)

1. Use the ASTC diagram to find the quadrants of the solutions (there will be two such quadrants).
2. For each of these quadrants, draw a sketch showing the line OP in that quadrant. On each sketch, mark the associated angle θ that lies in the interval $[0, 2\pi]$, and the acute angle ϕ between OP and the x-axis. (You can use the interval $[-\pi, \pi]$ instead of $[0, 2\pi]$, but you should use the same interval for each sketch.)



3. Find ϕ by applying the appropriate inverse trigonometric function to the equation $\sin\phi = |c|$, $\cos\phi = |c|$ or $\tan\phi = |c|$, as appropriate.
4. Use your sketches to find two values of θ in the interval $[0, 2\pi]$ (or in the interval $[-\pi, \pi]$) that are solutions of the equation.
5. If required, add multiples of 2π to obtain further solutions, or solutions in a different interval.

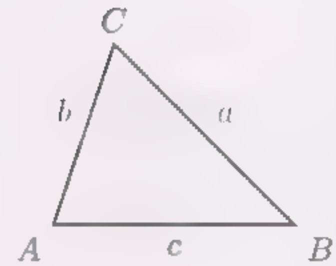
To solve an equation of the form $\sin\theta = c$, $\cos\theta = c$ or $\tan\theta = c$ by using a sketch graph

1. Sketch the graph of the relevant trigonometric function on the interval $[-\pi, \pi]$.
2. Find one solution of the equation by using the appropriate inverse trigonometric function, and mark it on your sketch.
3. Use the symmetry of the graph to find any other solutions in the interval $[-\pi, \pi]$ (usually there is one further such solution).
4. If required, add multiples of 2π to obtain further solutions, or solutions in a different interval.

Trigonometric rules

Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



When you use the sine rule to find an unknown angle, you usually obtain two possible angles, one acute and one obtuse. To determine which angle is correct, use additional information about the triangle.

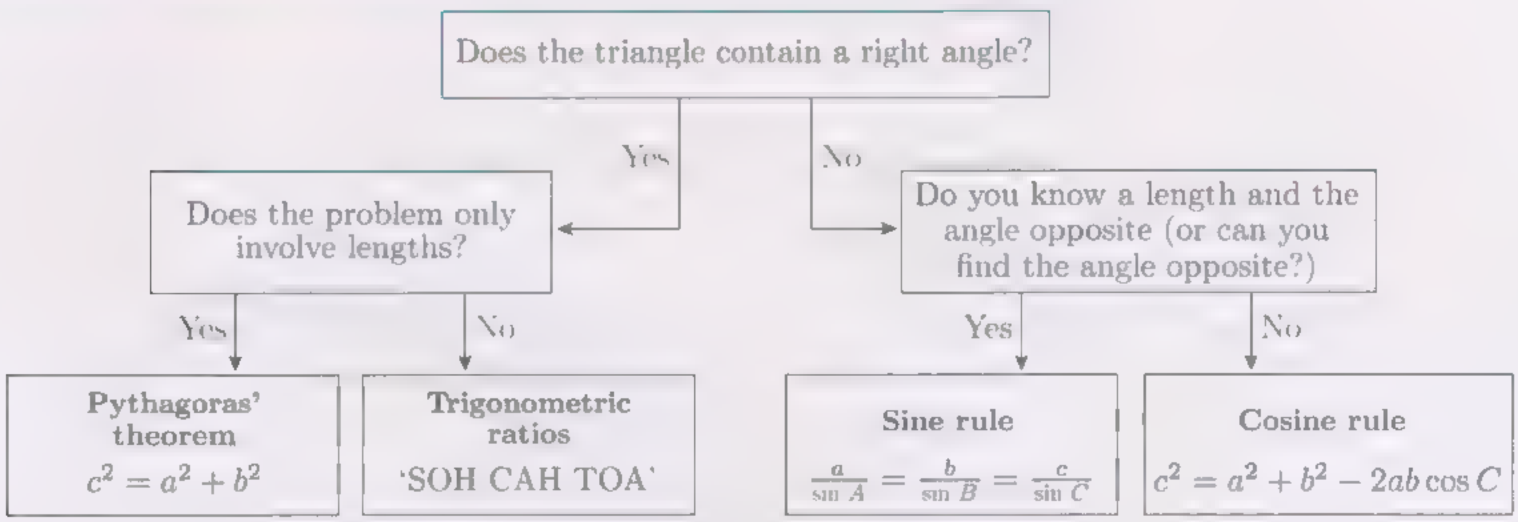
- If the obtuse angle leads to a total angle sum of more than 180°, then it is incorrect.
- Smaller angles are opposite shorter sides.

You may not have enough information to determine which angle is correct.

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

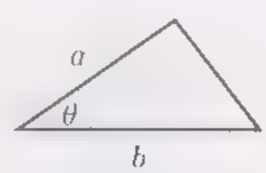
Choosing a method for finding unknown sides and angles in triangles



Area of a triangle

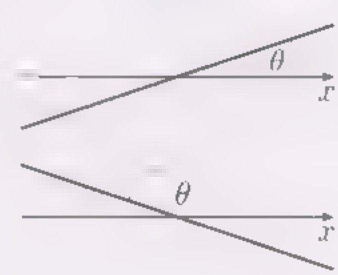
For a triangle with an angle θ between sides of lengths a and b,

$$\text{area} = \tfrac{1}{2}ab\sin\theta.$$



Angle of inclination of a line

The **angle of inclination** of a straight line is the angle that it makes with the x-axis, measured anticlockwise from the positive direction of the x-axis, when the line is drawn on axes *with equal scales*.



For any non-vertical straight line with angle of inclination θ,

$$\text{gradient} = \tan\theta.$$

Unit 5 Coordinate geometry and vectors

Coordinate geometry

Distance formula (for two or three dimensions)

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

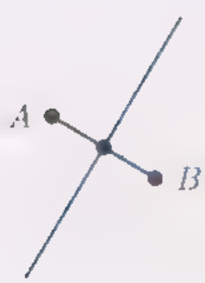
The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Midpoint formula (for two dimensions)

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

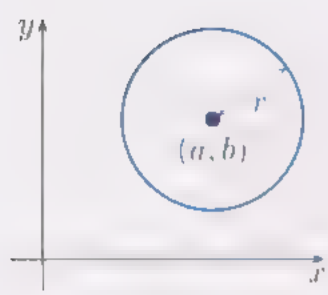


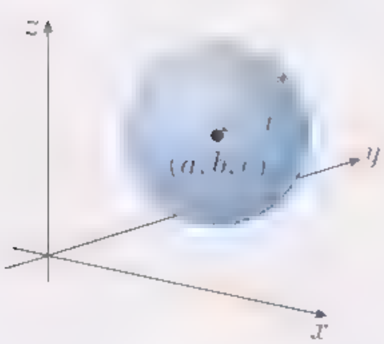
Perpendicular bisectors (for two dimensions)

The **perpendicular bisector** of a line segment is the line that is perpendicular to the line segment and divides it into two equal parts.

If A and B are points in the plane that do not lie on the same horizontal or vertical line, then the gradient of the perpendicular bisector of AB is

$$-\frac{1}{\text{gradient of } AB}.$$





Standard form of the equation of a circle or a sphere

The circle with centre (a, b) and radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

The sphere with centre (a, b, c) and radius r has equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

To find the equation of the circle passing through three points

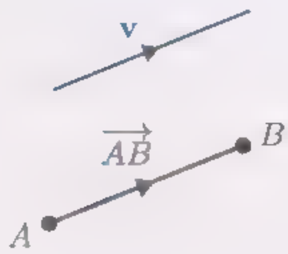
1. Find the equation of the perpendicular bisector of the line segment that joins any pair of the three points.
2. Find the equation of the perpendicular bisector of the line segment that joins a different pair of the three points.
3. Find the point of intersection of these two lines. This is the centre of the circle.
4. Find the radius of the circle, which is the distance from the centre to any of the three points.

Vectors

A **vector** is a quantity that has both a size (usually called **magnitude**) and a direction. The following quantities are vectors:

- **Displacement**, the position of one point relative to another.
- **Velocity**, the speed of an object together with its direction.

A **displacement vector** is a vector that represents displacement.



Scalars

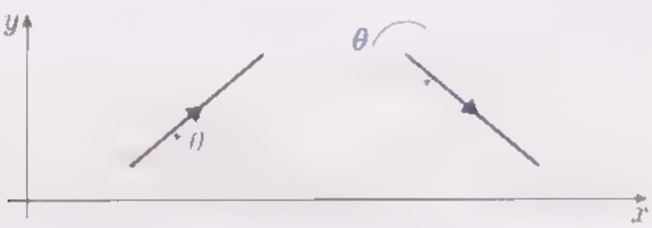
A **scalar** is a quantity that has size but no direction. The following quantities are scalars:

- Distance, the magnitude of displacement.
- Speed, the magnitude of velocity.
- Time, temperature and volume.

Direction of a vector

The **direction** of a vector is usually specified in one of the following two ways.

- As an angle measured anticlockwise from the positive direction of the x-axis to the direction of the vector.



- As a **bearing**: that is, the angle in degrees between 0° and 360° measured clockwise from north to the direction of the vector.

Heading and course of a ship or aircraft

For a moving ship or aircraft:

- Its **heading** is the direction in which it is pointing, given as a bearing.
- Its **course** is the direction in which it is actually moving.

These directions may be different, due to the effect of a current or wind.

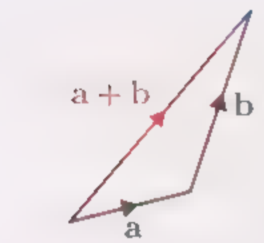
The actual velocity of a ship or aircraft is the resultant of the velocity that it would have if the water or air were still, and the velocity of the current or wind.

Vector algebra

Two vectors are **equal** if they have the same magnitude and the same direction.

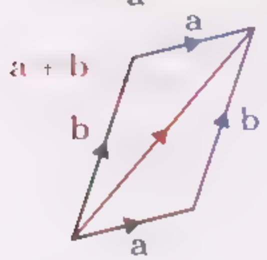
The zero vector

The **zero vector**, denoted by **0** (bold zero), is the vector whose magnitude is zero. It has no direction.



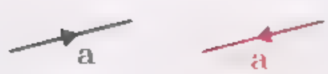
Triangle law for vector addition

To find the **sum** (**resultant**) of two vectors **a** and **b**, place the tail of **b** at the tip of **a**. Then **a** + **b** is the vector from the tail of **a** to the tip of **b**.



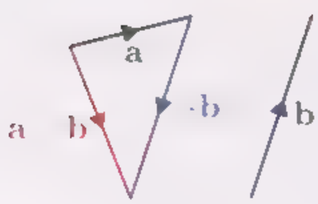
Parallelogram law for vector addition

To find the sum of two vectors **a** and **b**, place their tails together, and complete the resulting figure to form a parallelogram. Then **a** + **b** is the vector formed by the diagonal of the parallelogram, starting from the point where the tails of **a** and **b** meet.



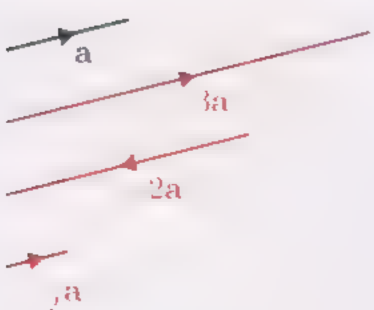
Negative of a vector

The **negative** of a vector **a**, denoted by −**a**, is the vector with the same magnitude as **a**, but the opposite direction.



Vector subtraction

To subtract **b** from **a**, add −**b** to **a**. That is, **a** − **b** = **a** + (−**b**).



Scalar multiple of a vector

Suppose that **a** is a vector. Then, for any non-zero real number m, the **scalar multiple** m**a** of **a** is the vector

- whose magnitude is $|m|$ times the magnitude of **a**
- that has the same direction as **a** if m is positive, and the opposite direction if m is negative.

Also, 0**a** = **0**. (That is, zero times the vector **a** is the zero vector.)

Properties of vector algebra

These properties hold for all vectors **a**, **b** and **c**, and all scalars m and n.

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}$
6. $(m+n)\mathbf{a}=m\mathbf{a}+n\mathbf{a}$
7. $m(n\mathbf{a})=(mn)\mathbf{a}$
8. $1\mathbf{a}=\mathbf{a}$

Representing vectors using component form

A **unit vector** is a vector with magnitude 1.

The **Cartesian unit vectors**, denoted by **i**, **j** and **k**, are the vectors of magnitude 1 in the directions of the x-, y- and z-axes, respectively.

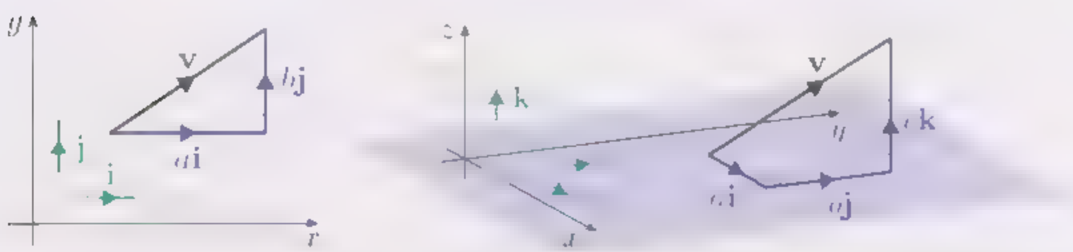
Component form of a vector

The **component form** of a two-dimensional vector **v** is the expression $a\mathbf{i} + b\mathbf{j}$, where $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

It can also be written as $\begin{pmatrix} a \\ b \end{pmatrix}$.

The **component form** of a three-dimensional vector **v** is the expression $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

It can also be written as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.



The **i-component** and **j-component** (or the x**-component** and y**-component**) of a vector **v** are the scalars a and b, respectively, in the component form $a\mathbf{i} + b\mathbf{j}$ of **v**. The components of a three-dimensional vector are referred to in a similar way.

A **column vector** is a vector written as a column, such as $\begin{pmatrix} a \\ b \end{pmatrix}$.

Vector algebra using component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, and m is a scalar, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} \qquad -\mathbf{a} = -a_1\mathbf{i} - a_2\mathbf{j}$$
$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} \qquad m\mathbf{a} = ma_1\mathbf{i} + ma_2\mathbf{j}.$$

In column notation, if $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and m is a scalar, then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \qquad -\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$$
$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \qquad m\mathbf{a} = \begin{pmatrix} ma_1 \\ ma_2 \end{pmatrix}.$$

The algebra of three-dimensional vectors is similar.

Converting vectors from component form to magnitude and direction, and vice versa

To find the magnitude of a two- or three-dimensional vector from its components

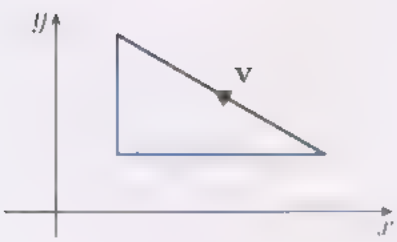
The two-dimensional vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ has magnitude

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$

The three-dimensional vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has magnitude

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}.$$

To find the magnitude and direction of a two-dimensional vector (not parallel to an axis) from its component form



1. Using the components, sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the x- and y-axes.
2. Use Pythagoras' theorem (or, equivalently, the formula above) to find the magnitude of the vector.
3. Use trigonometry to find an acute angle in the triangle.
4. Use this acute angle to work out the direction of the vector.

To find the component form of a two-dimensional vector (not parallel to an axis) from its magnitude and direction

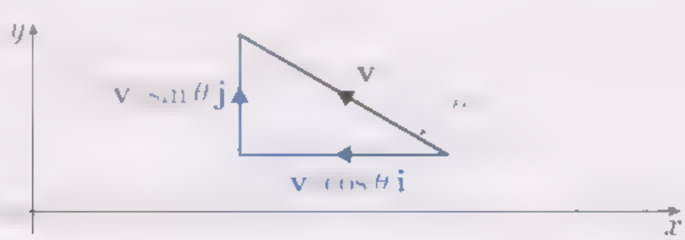
1. Using the magnitude and direction, sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the x- and y-axes.
2. Use trigonometry to find the magnitudes of the components.
3. Use the direction of the vector to find the signs of the components.

An alternative method is to use the formula below.

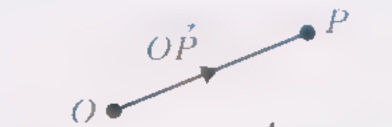
Component form of a two-dimensional vector in terms of its magnitude and the angle that it makes with the positive x-direction

If the two-dimensional vector $\mathbf{v}$ makes the angle θ with the positive x-direction, then

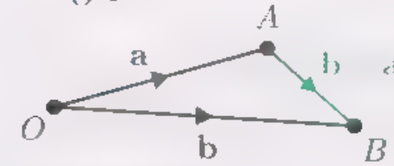
$$\mathbf{v} = |\mathbf{v}|\cos\theta\,\mathbf{i} + |\mathbf{v}|\sin\theta\,\mathbf{j}.$$



Position vectors



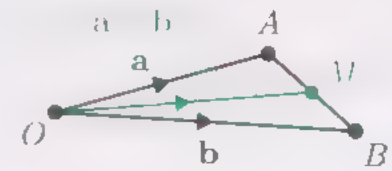
The **position vector** of a point P is the displacement vector $\overrightarrow{OP}$, where O is the origin.



If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, then

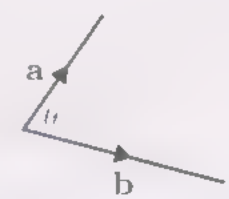
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

Midpoint formula in terms of position vectors



If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, then the midpoint of the line segment AB has position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Scalar product



The angle between two non-zero vectors is the angle θ in the range $0 \leq \theta \leq 180°$ between their directions when the vectors are placed tail to tail.

Scalar product of two vectors

The **scalar product** (or **dot product**) of the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, then $\mathbf{a}\cdot\mathbf{b} = 0$.

Properties of the scalar product

These properties hold for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and every scalar m.

1. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are non-zero. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a}\cdot\mathbf{b} = 0$, and vice versa.
2. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
3. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
4. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
5. $(m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(m\mathbf{b})$

Scalar product in terms of components

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2.$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

To find the angle between two vectors in component form

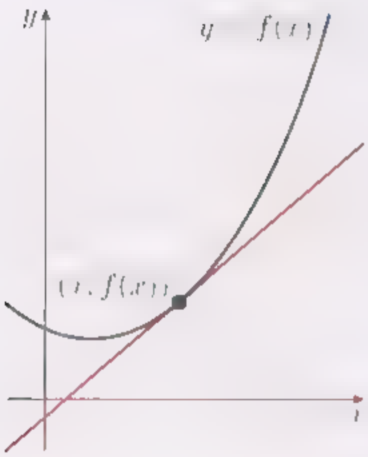
The angle θ between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|},$$

where $0 \leq \theta \leq 180°$.

Unit 6 Differentiation

Derivatives



The **tangent** to a curved graph at a particular point is the straight line that 'just touches' it. The **gradient** of the graph at that point is the gradient of the tangent.

A function f is **differentiable** at a particular value of x if its graph has a gradient at the point $(x, f(x))$.

The **derivative** (or **derived function**) of a function f is the function f' such that

$$f'(x) = \text{gradient of the graph of } f \text{ at the point } (x, f(x)).$$

The domain of f' consists of all the values in the domain of f at which f is differentiable.

The **derivative of f at x** is the value $f'(x)$.

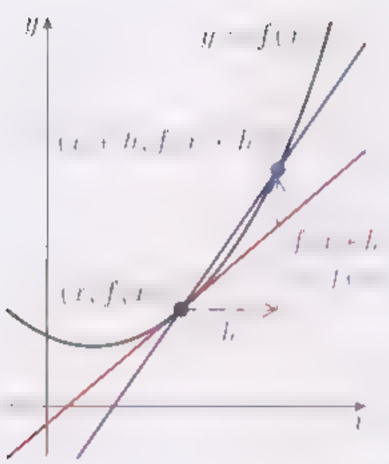
If $y = f(x)$, then $f'(x)$ is the **rate of change** of y with respect to x.

In **Lagrange notation**, the derivative of a function f is denoted by f'.

In **Leibniz notation**, if $y = f(x)$, then $f'(x)$ is denoted by $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}(f(x))$.

The quantity $\dfrac{dy}{dx}$ is called the **derivative of y with respect to x**.

Differentiation from first principles



The derivative f' of a function f is given by the equation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

where the notation '$\lim\limits_{h\to 0}$' means 'the **limit** as h tends to zero of' (the value approached as h approaches zero).

The **difference quotient for f at x** is the fraction in the equation above. It is the gradient of the line through the points $(x, f(x))$ and $(x+h, f(x+h))$.

Power functions

A **power function** is a function of the form $f(x) = x^n$, where n is a real number. The power function $f(x) = x^n$ has derivative

$$f'(x) = nx^{n-1}.$$

Constant multiple rule and sum rule for derivatives

If $k(x) = af(x)$, where a is a constant, then

$$k'(x) = af'(x).$$

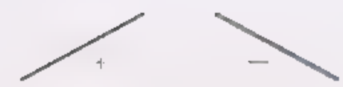
If $k(x) = f(x) + g(x)$, then

$$k'(x) = f'(x) + g'(x).$$

Increasing and decreasing parts of graphs

Increasing/decreasing criterion

If $f'(x)$ is positive for all x in an interval I, then f is increasing on I.
If $f'(x)$ is negative for all x in an interval I, then f is decreasing on I.



Stationary points

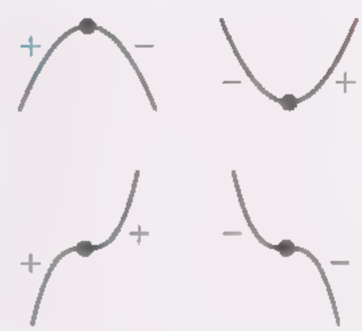
A **stationary point** of a function f is a value of x at which $f'(x) = 0$, or the corresponding point on the graph of f.

- A **local maximum** of f is a point where f takes a value larger than at any other point nearby.
- A **local minimum** of f is a point where f takes a value smaller than at any other point nearby.
- A **horizontal point of inflection** is a stationary point such that the graph is increasing on both sides, or decreasing on both sides.

A **turning point** is a local maximum or local minimum.

First derivative test (for determining the nature of a stationary point)

If there are open intervals immediately to the left and right of a stationary point of a function f such that



- $f'(x)$ is positive on the left interval and negative on the right interval, then the stationary point is a local maximum
- $f'(x)$ is negative on the left interval and positive on the right interval, then the stationary point is a local minimum
- $f'(x)$ is positive on both intervals or negative on both intervals, then the stationary point is a horizontal point of inflection.

You can apply the first derivative test directly (it can be helpful to construct a table of signs for $f'(x)$), or by choosing sample points.

To apply the first derivative test by choosing sample points

1. Choose two points (that is, two x-values) fairly close to the stationary point, one on each side.
2. Check that the function is differentiable at all points between the chosen points and the stationary point, and that there are no other stationary points between the chosen points and the stationary point.
3. Find the value of the derivative of the function at the two chosen points.

   - If the derivative is positive at the left chosen point and negative at the right chosen point, then the stationary point is a local maximum.
   - If the derivative is negative at the left chosen point and positive at the right chosen point, then the stationary point is a local minimum.
   - If the derivative is positive at both chosen points or negative at both chosen points, then the stationary point is a horizontal point of inflection.

Function continuous on an interval

A function is **continuous** on an interval if its graph has no 'breaks' (**discontinuities**) in the interval.

To find the greatest or least value of a function on an interval of the form $[a, b]$

(This strategy is valid when the function is continuous on the interval, and differentiable at all values in the interval except possibly the endpoints.)

1. Find the stationary points of the function.
2. Find the values of the function at any stationary points inside the interval, and at the endpoints of the interval.
3. Find the greatest or least of the function values found.

Higher derivatives

The **second derivative** (or **second derived function**) of a function f is the function obtained by differentiating f twice. The **third derivative**, the **fourth derivative**, and so on, of f are defined in a similar way.

A function f is **n-times differentiable** at a value x if its nth derivative is defined at x.

Every polynomial function (with domain $\mathbb{R}$) is differentiable infinitely many times at every value of x.

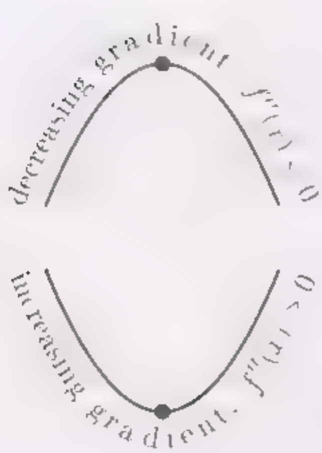
(In fact, as stated in Unit 11, all polynomial, rational, trigonometric, exponential and logarithmic functions, and all constant multiples, sums, differences, products, quotients and composites of these, are differentiable infinitely many times at every value of x in their domains.)

Concave up and concave down

A graph is **concave up** on an interval if the tangents to the graph on that interval lie below the graph. *Mnemonic*: Concave up, like a cup.

A graph is **concave down** on an interval if the tangents to the graph on that interval lie above the graph. *Mnemonic*: Concave down, like a frown.

A **point of inflection** is a point where a graph changes from concave up to concave down or vice versa.



Concave up/concave down criterion

If $f''(x)$ is positive for all x in an interval I, then f is concave up on I.

If $f''(x)$ is negative for all x in an interval I, then f is concave down on I.

Second derivative test (for determining the nature of a stationary point)

If, at a stationary point of a function, the value of the second derivative of the function is

- negative, then the stationary point is a local maximum ☹
- positive, then the stationary point is a local minimum. ☺

Displacement, velocity and acceleration

Suppose that an object is moving along a straight line. If t is the time that has elapsed since some chosen point in time, and s, v and a are the displacement, velocity and acceleration of the object, respectively, then

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad a = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}.$$

(Time, displacement, velocity and acceleration can be measured in any suitable units, as long as they are consistent.)

Note that the vector quantities displacement, velocity and acceleration are one-dimensional here, so they are represented as scalars, with a plus or minus sign to indicate the direction.

Total cost and marginal cost

The **total cost** of making a particular quantity of a product includes costs that are the same no matter how much of the product is made, and costs that depend on how much of the product is made.

The **unit cost** (or **average cost**) of making a particular quantity of a product is the cost per unit of making that quantity of the product.
It is the total cost of making that quantity of the product divided by the number of units of the product made.

The **marginal cost** of making a product is the cost per unit of making more of the product, when a particular quantity of the product is already being made.

The total cost, unit cost and marginal cost are all functions of the quantity of the product made.

The marginal cost is the derivative of the total cost with respect to the quantity of the product made.

Unit 7 Differentiation methods and integration

For standard derivatives and standard integrals, see page 7.

Differentiation rules

Product rule

Lagrange notation If $k(x) = f(x)g(x)$, then

$$k'(x) = f(x)g'(x) + g(x)f'(x).$$

Leibniz notation If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Informally

$$\begin{pmatrix}\text{derivative}\\ \text{of product}\end{pmatrix} = (\text{first}) \times \begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix} + (\text{second}) \times \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix}.$$

Quotient rule

Lagrange notation If $k(x) = f(x)/g(x)$, then

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Leibniz notation If $y = u/v$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Informally

$$\begin{pmatrix}\text{derivative}\\ \text{of quotient}\end{pmatrix} = \frac{(\text{bottom}) \times \begin{pmatrix}\text{derivative}\\ \text{of top}\end{pmatrix} - (\text{top}) \times \begin{pmatrix}\text{derivative}\\ \text{of bottom}\end{pmatrix}}{(\text{bottom})^2}$$

Chain rule

Lagrange notation If $k(x) = g(f(x))$, then

$$k'(x) = g'(f(x))f'(x).$$

Leibniz notation If y is a function of u, where u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Derivative of a function of a linear expression

If $k(x) = f(ax + b)$, where a and b are constants, then

$$k'(x) = af'(ax + b).$$

In particular, if $k(x) = f(ax)$, where a is a constant, then

$$k'(x) = af'(ax).$$

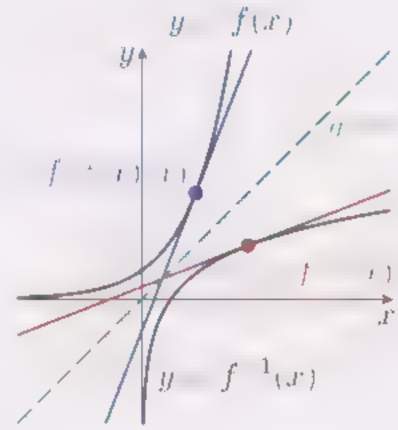
Inverse function rule

Lagrange notation If f is a function with inverse function f^{-1}, then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Leibniz notation If y is an invertible function of x, then

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$



Choosing a method for differentiating a function

1. Is it a standard function (is its derivative given on page 7)?
2. Can you use the constant multiple rule and/or sum rule?
3. Can you rewrite it to make it easier to differentiate?
   (Multiplying out brackets may help.)
4. Is it of the form $f(ax)$ or $f(ax + b)$, where a and b are constants?
   If so, use the rule for differentiating a function of a linear expression.
5. Can you use the product rule?
   Is it of the form $f(x) =$ something $\times$ something?
6. Can you use the quotient rule?
   Is it of the form $f(x) =$ something/something?
7. Can you use the chain rule?
   Is it of the form $f(x) =$ a function of something?

When you use a differentiation rule, you usually have to find the derivatives of simpler functions. Apply the checklist above to each of these simpler functions in turn.

Optimisation problems

An **optimisation problem** involves identifying the best possible option from a choice of suitable possibilities. A **maximisation** or **minimisation problem** involves identifying the circumstances under which the maximum or minimum value, respectively, of a quantity is obtained.

To solve an optimisation problem

1. Identify the quantity that you can change, and represent it by a variable, noting the possible values that it can take. Identify the quantity to be maximised or minimised, and represent it by a variable. These variables are the independent and dependent variables, respectively.
2. Find a formula for the dependent variable in terms of the independent variable.
3. Use the techniques of differential calculus to find the value of the independent variable that gives the maximum/minimum value of the dependent variable. (The strategy for finding the greatest or least value of a function on an interval of the form $[a, b]$, on page 46, is often useful.)
4. Interpret your answer in the context of the problem.

Antiderivatives and indefinite integrals

A **continuous function** is a function whose graph has no 'breaks' (discontinuities) over its whole domain.

An **antiderivative** of a function f is any specific function whose derivative is f.

The **indefinite integral** of a continuous function f is the general function obtained by adding an **arbitrary constant** c (the **constant of integration**) to the formula for an antiderivative of f. It describes the complete family of antiderivatives of f.

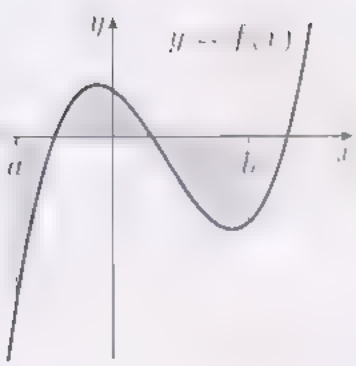
An **indefinite integral** of a function f that is not continuous is a general function obtained by adding an arbitrary constant to the formula for an antiderivative of f. (It can be used to provide the indefinite integral of any continuous function that has the same rule as f.)

Integration (or **antidifferentiation**) is the process of finding an antiderivative of a function. It is the reverse of differentiation.

Unit 8 Integration methods

Signed areas and definite integrals

Signed areas



The **signed area** of a region that lies entirely above or entirely below the x-axis is its area with a plus or minus sign according to whether it lies above or below the x-axis, respectively.

The **signed area** of a collection of such regions is the sum of the signed areas of the individual regions.

The **signed area between** the graph of a continuous function f and the x-axis **from** $x = a$ **to** $x = b$, where a and b are numbers in its domain, is

- the signed area of the collection of regions described, if $a \leq b$;
- the negative of this signed area, if $b \leq a$.

Definite integrals

The **definite integral** of a continuous function f from a to b, where a and b are numbers in its domain, is the signed area between the graph of f and the x-axis from $x = a$ to $x = b$. It is denoted by

$$\int_a^b f(x)\,dx$$

- The numbers a and b are the **lower** and **upper limits of integration**, respectively.
- The expression $f(x)$ is the **integrand**.
- The variable x is a **dummy variable**: you can change its name to any other variable name without affecting the value of the definite integral.

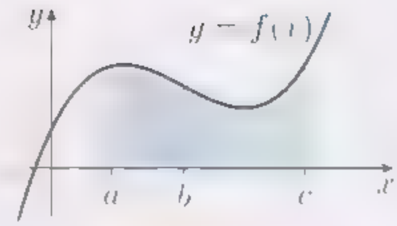
Standard properties of definite integrals

These properties hold for all numbers a, b and c in the domain of a continuous function f.

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

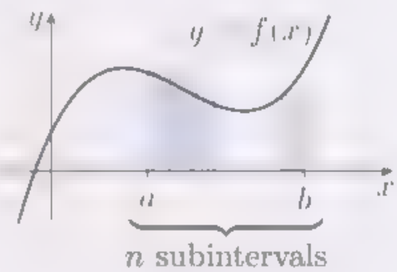


Algebraic definition of a definite integral

The **definite integral** of a continuous function f from $x = a$ to $x = b$, where a and b are numbers in its domain, is given by

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\Big(f(a+0w) + f(a+1w) + f(a+2w) + \cdots + f(a+(n-1)w)\Big)w,$$

where $w = (b-a)/n$.



Theorem

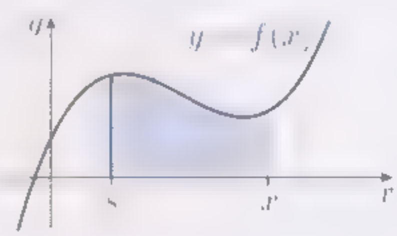
Suppose that f is a continuous function, and s is any number in its domain. Let A be the function with the same domain as f and rule

$$A(x) = \int_s^x f(t)\,\mathrm{d}t.$$

Then A is an antiderivative of f.

It follows that: every continuous function has an antiderivative.

$A(x)$ = signed area from s to x



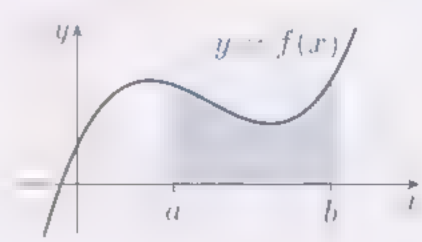
Fundamental theorem of calculus

Suppose that f is a continuous function whose domain contains the numbers a and b, and that F is an antiderivative of f. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

In **square bracket notation**, $F(b) - F(a)$ is denoted by $\big[F(x)\big]_a^b$.

$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$



Notation for indefinite integrals

The indefinite integral of $f(x)$ is denoted by $\int f(x)\,\mathrm{d}x$.

The expression $f(x)$ in this notation is the **integrand**.

Integration rules

Constant multiple rule and sum rule for integrals

For definite integrals

$$\int_a^b k\,f(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x, \quad \text{where } k \text{ is a constant}$$

$$\int_a^b \big(f(x)+g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

For indefinite integrals

$$\int k\,f(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x, \quad \text{where } k \text{ is a constant}$$

$$\int \big(f(x)+g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

For the square bracket notation

$$\big[kF(x)\big]_a^b = k\big[F(x)\big]_a^b, \quad \text{where } k \text{ is a constant}$$

$$\big[F(x)+G(x)\big]_a^b = \big[F(x)\big]_a^b + \big[G(x)\big]_a^b$$

Integration by substitution

1. Recognise that the integrand is of the form

   $$f(\text{something}) \times \text{the derivative of the something},$$

   where f is a function that you can integrate.
2. Set the something equal to u, and find $\mathrm{d}u/\mathrm{d}x$.
3. Hence write the integral in the form

   $$\int f(u)\,\mathrm{d}u,$$

   by using the fact that $\displaystyle\int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$.
4. Do the integration.
5. Substitute back for u in terms of x.

For a definite integral, in step 3 also change the limits of integration, which are values of x, to the corresponding values of u. Then omit step 5.

Indefinite integral of a function of a linear expression

Suppose that f is a function with antiderivative F.

If a and b are constants with a non-zero, then

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + c.$$

In particular, if a is a non-zero constant, then

$$\int f(ax)\,\mathrm{d}x = \frac{1}{a}F(ax) + c.$$

Integration by parts

Lagrange notation $\displaystyle\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx.$

Here G is an antiderivative of g.

Alternative version $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

Leibniz notation $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Informally

$$\begin{pmatrix}\text{integral}\\ \text{of product}\end{pmatrix} = (\text{first}) \times \begin{pmatrix}\text{antiderivative}\\ \text{of second}\end{pmatrix}$$

$$- \text{integral of}\left(\begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix} \times \begin{pmatrix}\text{antiderivative}\\ \text{of second}\end{pmatrix}\right).$$

For definite integrals $\displaystyle\int_a^b f(x)g(x)\,dx = \Big[f(x)G(x)\Big]_a^b - \int_a^b f'(x)G(x)\,dx.$

Here G is an antiderivative of g.

Choosing a method for finding an integral

- Is it a standard integral? Consult the table on page 7.
- Can you rearrange the integral to express it as a sum of constant multiples of simpler integrals?
- Is the integrand of the form $f(ax)$ or $f(ax+b)$, where a and b are constants and f is a function that you can integrate? If so, use the rule for integrating a function of a linear expression, or use integration by substitution, with $u = ax$ or $u = ax + b$ as appropriate.
- Can the integrand be written in the form

  f(something) $\times$ the derivative of the something,

  where f is a function that you can integrate? (To obtain this form you might need to multiply by a constant inside the integral, and divide by the same constant outside.) If so, use integration by substitution. Start by setting the 'something' equal to u.
- Is the integrand of the form $f(x)g(x)$, where f is a function that becomes simpler when differentiated, and g is a function that you can integrate? (For example, $f(x)$ might be x, or x^2, or any polynomial expression in x, or it might be $\ln x$.) If so, try integration by parts.
- If the integrand contains trigonometric functions, can you use trigonometric identities to rewrite it in a form that's easier to integrate?

To integrate an expression of the form $e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$

Integrate by parts twice, to obtain an equation that expresses the original integral in terms of itself, then rearrange this to find the original integral.

Unit 9 Matrices

Matrices and matrix operations

A **matrix** is a rectangular array of numbers. An **element** (or **entry**) of a matrix is one of the numbers in it. The element in row i and column j of a matrix $\mathbf{A}$ is denoted by a_{ij}.

A **row** of a matrix is a horizontal line of numbers in it, and a **column** of a matrix is a vertical line of numbers in it. An $m \times n$ matrix, or matrix of **size** $m \times n$, has m rows and n columns.

A **square matrix** has the same number of rows as columns.

A **vector** is a matrix with one column. The **components** of a vector are its elements. An n-**dimensional** vector has n components.

Zero matrix

A **zero matrix** is a matrix each of whose elements is zero.

Negative of a matrix

The **negative** of a matrix $\mathbf{A}$, denoted by $-\mathbf{A}$, is the matrix obtained by changing the sign of each element of $\mathbf{A}$.

Matrix addition and subtraction

If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size, then $\mathbf{A}+\mathbf{B}$ is the matrix obtained by adding the corresponding elements of $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{A}-\mathbf{B}$ is the matrix obtained by subtracting from each element of $\mathbf{A}$ the corresponding element of $\mathbf{B}$. Two matrices of different sizes cannot be added or subtracted.

Scalar multiplication of a matrix

If $\mathbf{A}$ is a matrix and k is a number, then $k\mathbf{A}$ is the matrix obtained by multiplying each element of $\mathbf{A}$ by k.

Properties of matrix addition and scalar multiplication

These properties hold for all matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the same size, and all scalars m and n.

1. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
2. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
3. $\mathbf{A}+\mathbf{0}=\mathbf{A}$
4. $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$
5. $m(\mathbf{A}+\mathbf{B})=m\mathbf{A}+m\mathbf{B}$
6. $(m+n)\mathbf{A}=m\mathbf{A}+n\mathbf{A}$
7. $m(n\mathbf{A})=(mn)\mathbf{A}$
8. $1\mathbf{A}=\mathbf{A}$

In properties 3 and 4, $\mathbf{0}$ is the zero matrix of the same size as $\mathbf{A}$.

Matrix multiplication

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix, then $\mathbf{AB}$ is the $m \times p$ matrix in which, for each i and j, the entry in row i and column j is obtained by multiplying each element in the ith row of $\mathbf{A}$ by the corresponding element in the jth column of $\mathbf{B}$, and adding the results.

$$\begin{array}{ccccc}
 & & \text{column } j & & \text{column } j \\
\text{row } i & \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} & \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} & = & \begin{pmatrix} \cdots & \cdot & \cdots \end{pmatrix} \; \text{row } i
\end{array}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

If the number of columns of a matrix $\mathbf{A}$ is not equal to the number of rows of a matrix $\mathbf{B}$, then $\mathbf{AB}$ is not defined.

Properties of matrix multiplication

These properties hold for all matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for which the products and sums mentioned are defined.

1. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2. $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$, for any scalar k
3. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
4. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

For matrices $\mathbf{A}$ and $\mathbf{B}$:

- The product $\mathbf{AB}$ may exist but the product $\mathbf{BA}$ may not.
- If both $\mathbf{AB}$ and $\mathbf{BA}$ exist, they are usually not equal.

Matrix powers

The **square** $\mathbf{A}^2$ of a square matrix $\mathbf{A}$ is the matrix product $\mathbf{AA}$.

The nth **power** $\mathbf{A}^n$ of a square matrix $\mathbf{A}$ is is obtained by multiplying together n copies of $\mathbf{A}$. For example, $\mathbf{A}^3 = \mathbf{AAA}$.

Matrix inverses and determinants

Identity matrices

An **identity matrix** is a square matrix $\mathbf{I}$ of the form $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.

If $\mathbf{I}$ is an identity matrix, then

- for any matrix $\mathbf{A}$ for which the product $\mathbf{AI}$ is defined, $\mathbf{AI} = \mathbf{A}$
- for any matrix $\mathbf{A}$ for which the product $\mathbf{IA}$ is defined, $\mathbf{IA} = \mathbf{A}$.

Inverse of a matrix

A square matrix $\mathbf{A}$ is **invertible** if there is a matrix $\mathbf{B}$ of the same size such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix. For each square matrix $\mathbf{A}$ there is at most one such matrix $\mathbf{B}$; if there is such a matrix $\mathbf{B}$, then it is called the **inverse** of $\mathbf{A}$ and denoted by $\mathbf{A}^{-1}$.

So, if $\mathbf{A}$ is an invertible matrix, then

$$\mathbf{AA}^{-1} = \mathbf{I} \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

A square matrix is **non-invertible** if it has no inverse.

Determinant of a matrix

The **determinant** of a square matrix $\mathbf{A}$ is a number calculated from its elements. It is denoted by $\det \mathbf{A}$.

- If $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible.
- If $\det \mathbf{A} = 0$, then $\mathbf{A}$ is not invertible.

Determinant and inverse of a 2×2 matrix

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

- $\det \mathbf{A} = ad - bc$
- $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided that $ad - bc \neq 0$

Systems of linear equations

A **system of linear equations** is a set of linear equations, each in the same set of unknowns, that apply simultaneously.

A **solution** of the system is an assignment of values to the unknowns that makes all the equations true simultaneously.

Matrix form of a system of linear equations

A system of n linear equations in n unknowns can be written as a single matrix equation

$$\mathbf{Ax} = \mathbf{b},$$

where

- $\mathbf{A}$ is an $n \times n$ matrix called the **coefficient matrix**
- $\mathbf{x}$ is an n-dimensional vector whose components are the unknowns
- $\mathbf{b}$ is an n-dimensional vector.

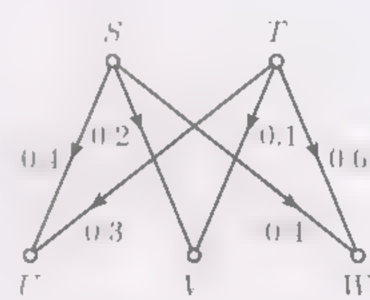
The system $\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$ has matrix form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$.

Solutions of a system of linear equations

- If $\det \mathbf{A} \neq 0$, then the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution, given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.
- If $\det \mathbf{A} = 0$, then the system $\mathbf{Ax} = \mathbf{b}$ has no solution or infinitely many solutions.

Matrices and networks

A network with n input nodes and m output nodes, in which all flow is directly from input nodes to output nodes, can be represented by an $m \times n$ matrix $\mathbf{A}$ in which, for each i and j, the element a_{ij} is the proportion of the flow from the jth input node that goes to the ith output node. (There is an example in the margin.)



$$\begin{array}{c} \\ U \\ V \\ W \end{array}\begin{array}{c} \begin{array}{cc} S & T \end{array} \\ \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \\ 0.4 & 0.6 \end{pmatrix} \end{array} = \mathbf{A}$$

If the n input values are represented by an n-dimensional vector $\mathbf{x}$, and the corresponding m output values are represented by an m-dimensional vector $\mathbf{y}$, then $\mathbf{y} = \mathbf{Ax}$.

Combining networks

If the output nodes of one network are also the input nodes of a second network, then the combined network is equivalent to a simpler network in which flow is directly from the input nodes of the first network to the output nodes of the second network.

This simpler network is represented by the matrix $\mathbf{BA}$, where $\mathbf{A}$ and $\mathbf{B}$ are the matrices representing the first and second networks respectively.

Unit 10 Sequences and series

Sequences

A **sequence** is a list of numbers, called its **terms**.

A **finite sequence** has a finite number of terms.

An **infinite sequence** has an infinite number of terms.

The notation (a_n) denotes the infinite sequence $a_1, a_2, a_3, \ldots$.
The variable n is the **index variable**.

The first term of a sequence is assumed to have subscript 1, unless otherwise indicated.

A **constant sequence** is a sequence in which every term has the same value.

Closed form for a sequence

A **closed form** for a sequence is a formula that defines the general term a_n as an expression involving the subscript n. To specify a sequence using a closed form, two pieces of information are needed:

- the closed form
- the range of values for the subscript n.

Recurrence system for a sequence

A (**first-order**) **recurrence relation** for a sequence is an equation that defines each term a_n other than the first as an equation involving the previous term a_{n-1}. To specify a sequence using a **recurrence system**, three pieces of information are needed:

- the value of the first term
- the recurrence relation
- the range of values for the subscript n.

Arithmetic sequences

An **arithmetic sequence** is a sequence in which each term (apart from the first) is obtained by adding a fixed number, called the **common difference**, to the previous term.

The arithmetic sequence (x_n) with first term a and common difference d has recurrence system

$$x_1 = a, \quad x_n = x_{n-1} + d \quad (n = 2, 3, 4, \ldots)$$

and closed form

$$x_n = a + (n-1)d \quad (n = 1, 2, 3, \ldots).$$

An **arithmetic series** is an expression obtained by adding consecutive terms of an arithmetic sequence.

Geometric sequences

A **geometric sequence** is a sequence in which each term (apart from the first) is obtained by multiplying the previous term by a fixed number, called the **common ratio**.

The geometric sequence (x_n) with first term a and common ratio r has recurrence system

$$x_1 = a, \qquad x_n = rx_{n-1} \quad (n = 2, 3, 4, \ldots)$$

and closed form

$$x_n = ar^{n-1} \quad (n = 1, 2, 3, \ldots).$$

A **geometric series** is an expression obtained by adding consecutive terms of a geometric sequence.

Long-term behaviour of infinite sequences

Types of long-term behaviour

A sequence (x_n)

- is **increasing** if $x_n < x_{n+1}$ for each pair of successive terms x_{n+1} and x_n
- is **decreasing** if $x_n > x_{n+1}$ for each pair of successive terms x_{n+1} and x_n
- is **bounded** if all the terms lie within some interval $[-A, A]$, where A is a fixed positive number
- is **unbounded** if there is no fixed value of A, however large, for which all the terms lie within the interval $[-A, A]$
- **converges** to the **limit** L if its terms approach L more and more closely, so that eventually they lie within any interval $[L-h, L+h]$, no matter how small the positive number h is taken to be. We say that x_n **tends to** L **as** n **tends to infinity**, and write

$$x_n \to L \text{ as } n \to \infty \quad \text{or} \quad \lim_{n\to\infty} x_n = L$$

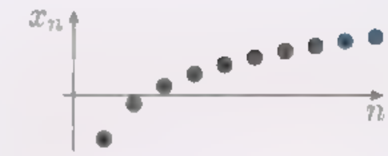
- **tends to infinity** if its terms increase without limit, so that eventually they lie in any interval (A, ∞), no matter how large the positive number A is taken to be. We say that x_n **tends to infinity as** n **tends to infinity**, and write

$$x_n \to \infty \text{ as } n \to \infty$$

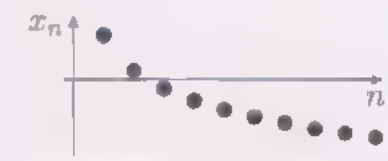
- **tends to minus infinity** if its terms decrease without limit, so that eventually they lie in any interval $(-\infty, -A)$, no matter how large the positive number A is taken to be. We say that x_n **tends to minus infinity as** n **tends to infinity**, and write

$$x_n \to -\infty \text{ as } n \to \infty.$$

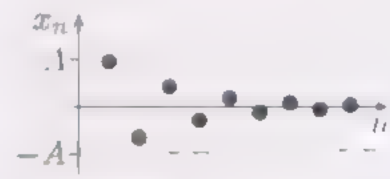
Some examples are shown in the margin.



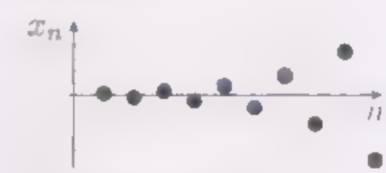
Increasing



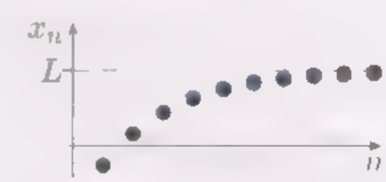
Decreasing



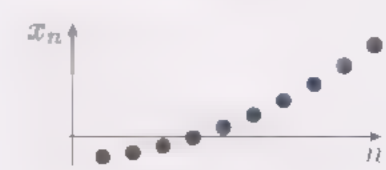
Bounded



Unbounded

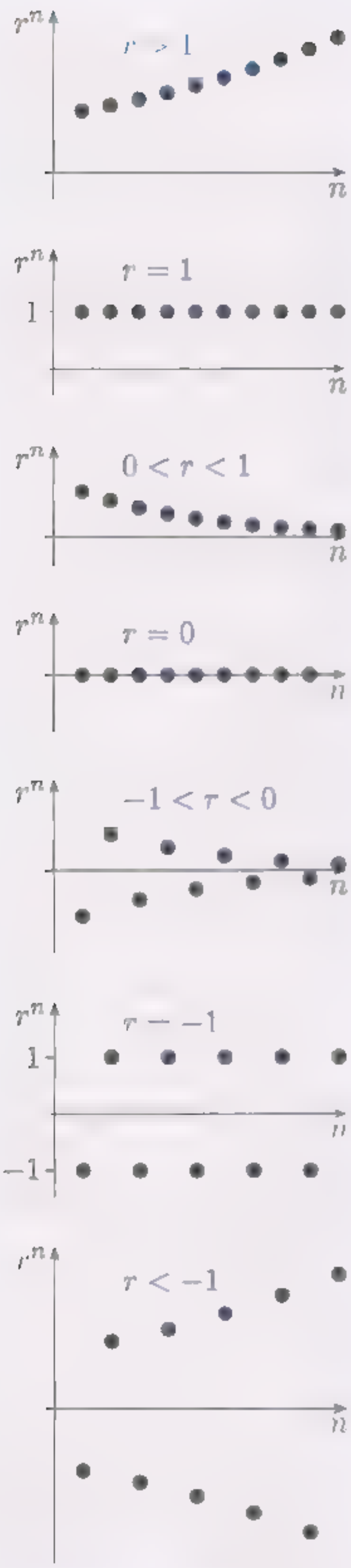


Converging to L



Tending to infinity

Tending to minus infinity



Behaviour of (r^n)

Long-term behaviour of arithmetic sequences

Suppose that (x_n) is an arithmetic sequence with common difference d.

- If $d > 0$, then (x_n) is increasing and $x_n \to \infty$ as $n \to \infty$.
- If $d < 0$, then (x_n) is decreasing and $x_n \to -\infty$ as $n \to \infty$.
- If $d = 0$, then (x_n) is constant.

Long-term behaviour of the sequence (r^n)

Value of r	Behaviour of (r^n)
$r > 1$	Increasing, $r^n \to \infty$ as $n \to \infty$
$r = 1$	Constant: $1, 1, 1, \ldots$
$0 < r < 1$	Decreasing, $r^n \to 0$ as $n \to \infty$
$r = 0$	Constant: $0, 0, 0, \ldots$
$-1 < r < 0$	Alternates in sign, $r^n \to 0$ as $n \to \infty$
$r = -1$	Alternates between -1 and 1
$r < -1$	Alternates in sign, unbounded

Effects of multiplying each term by a constant

Suppose that (x_n) is a sequence and c is a constant.

- If $c \neq 0$ and (x_n) $\left\{\begin{array}{l}\text{is constant}\\ \text{alternates in sign}\\ \text{is bounded}\\ \text{is unbounded}\\ \text{tends to } 0\end{array}\right\}$, then so is/does (cx_n).
- If $c > 0$ and (x_n) $\left\{\begin{array}{l}\text{is increasing}\\ \text{is decreasing}\\ \text{tends to } \infty\\ \text{tends to } -\infty\end{array}\right\}$, then so is/does (cx_n).
- If $c < 0$ and (x_n) $\left\{\begin{array}{l}\text{is increasing}\\ \text{is decreasing}\\ \text{tends to } \infty\\ \text{tends to } -\infty\end{array}\right\}$, then (cx_n) $\left\{\begin{array}{l}\text{is decreasing}\\ \text{is increasing}\\ \text{tends to } -\infty\\ \text{tends to } \infty\end{array}\right\}$.
- If $x_n \to L$ as $n \to \infty$, then $cx_n \to cL$ as $n \to \infty$.

Effects of adding a constant to each term

Suppose that (x_n) is a sequence and a and L are constants.

- If $x_n \to L$ as $n \to \infty$, then $x_n + a \to L + a$ as $n \to \infty$.
- If $x_n \to \infty$ as $n \to \infty$, then $x_n + a \to \infty$ as $n \to \infty$.
- If $x_n \to -\infty$ as $n \to \infty$, then $x_n + a \to -\infty$ as $n \to \infty$.

Series

A **series** is an expression obtained by adding consecutive terms of a sequence.

Sums of series

The **sum** of a finite series is the number obtained by adding up all the terms in the series.

The nth **partial sum** s_n of an infinite series $a_1 + a_2 + a_3 + \cdots$ is the number obtained by adding up the first n terms: $s_n = a_1 + a_2 + \cdots + a_n$.

An infinite series $a_1 + a_2 + a_3 + \cdots$ has **sum** s if its sequence of partial sums (s_n) converges to the limit s.

For formulas for sums of standard finite series, see page 8.

Sigma notation

- The finite sum $x_p + x_{p+1} + \cdots + x_q$ is denoted by $\displaystyle\sum_{n=p}^{q} x_n$.
- The infinite sum $x_p + x_{p+1} + \cdots$ is denoted by $\displaystyle\sum_{n=p}^{\infty} x_n$.

The variable n is the **index variable.** It is a dummy variable.
The numbers p and q are the **lower** and **upper limits**, respectively.

Rules for manipulating finite series in sigma notation

$$\sum_{k=p}^{q} c x_k = c \sum_{k=p}^{q} x_k \quad \text{(where } c \text{ is a constant)}$$

$$\sum_{k=p}^{q} (x_k + y_k) = \sum_{k=p}^{q} x_k + \sum_{k=p}^{q} y_k$$

$$\sum_{k=p}^{q} x_k = \sum_{k=1}^{q} x_k - \sum_{k=1}^{p-1} x_k \quad \text{(where } 1 < p \leq q\text{)}$$

These rules also apply when q is replaced by ∞, provided that each series involved has a sum.

The binomial theorem

Pascal's Triangle

1

1 1

1 3 3 1

1 5 10 16 5 1

1 6 15 20 15

For any natural number n,

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + \cdots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$$
$$= \sum_{k=0}^{n} {}^nC_k\, a^{n-k} b^k,$$

where the **binomial coefficient** ${}^nC_k$ is given by

$${}^nC_k = \frac{n!}{k!\,(n-k)!} \quad \text{for } k = 0, 1, 2, \ldots, n.$$

$\left(\text{For } 0 < k \leq n, \text{ alternatively } {}^nC_k = \dfrac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.\right)$

Here $n!$ denotes n **factorial**. If $n \geq 1$ then $n! = 1 \times 2 \times \cdots \times n$; $0! = 1$.

Unit 11 Taylor polynomials

Taylor polynomials

Let f be a function that is n-times differentiable at a point a. The **Taylor polynomial of degree n about a for f** is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The point a is called the **centre** of the Taylor polynomial.

When $a = 0$, the Taylor polynomial becomes

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

A Taylor polynomial of degree n can be denoted by $p_n(x)$.

The function f and the Taylor polynomial p of degree n about a for f have the same value at $x = a$, and the first, second, third, $\dots$, nth derivatives of f have the same values at $x = a$ as the corresponding derivatives of p.

(Note that we use **point** to mean 'number'.)

Taylor series

Let f be a function that is differentiable infinitely many times at a point a. The **Taylor series about a for f** is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The point a is called the **centre** of the Taylor series.

When $a = 0$, the Taylor series becomes

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The **Maclaurin series** for a function f is its Taylor series about 0.

Validity of Taylor series

A Taylor series about a point a for a function f is **valid** at a point x if for that value of x the Taylor series has sum $f(x)$.

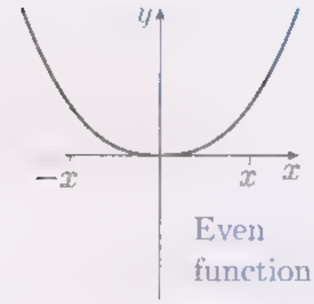
An **interval of validity** for a Taylor series about a point a for a function f is an interval of points for which the Taylor series is valid.

Standard Taylor series

For some standard Taylor series about 0, see page 8.

The **binomial series** is the standard Taylor series about 0 for $(1+x)^\alpha$.

Even and odd functions

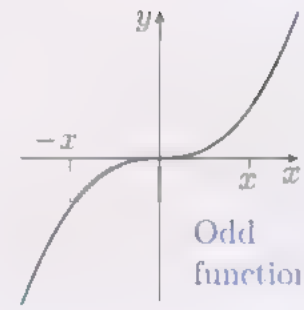


A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

The graph of an even function is unchanged under reflection in the y-axis.

A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

The graph of an odd function is unchanged by a rotation through a half-turn about the origin.

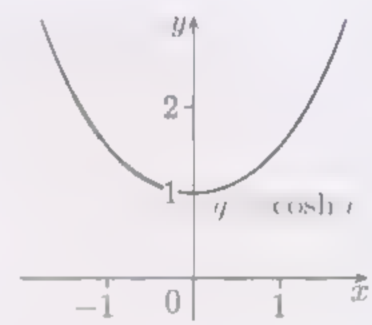


The cosine function is an even function.
The sine function is an odd function.

Taylor polynomials and series about 0 for even and odd functions

- A Taylor polynomial or Taylor series about 0 for an even function contains terms in even powers of x only.
- A Taylor polynomial or Taylor series about 0 for an odd function contains terms in odd powers of x only.

Hyperbolic functions



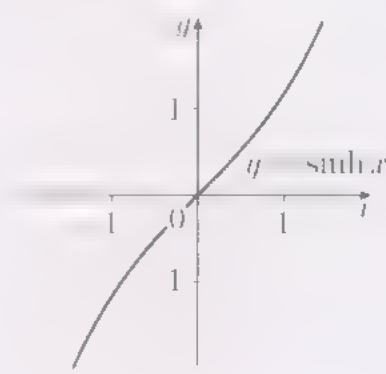
The **hyperbolic cosine function** is given by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

The **hyperbolic sine function** is given by

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

The hyperbolic cosine function is an even function.
The hyperbolic sine function is an odd function.



The Taylor series about 0 for these functions are:

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots, \quad \text{for } x \in \mathbb{R}$$

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots, \quad \text{for } x \in \mathbb{R}.$$

Using Taylor polynomials for approximation

To find an approximation for $f(x)$, for a particular function f at a particular point x

1. Find a Taylor series for f about a suitable point a close to x, and check that it is valid at x.
2. Truncate the Taylor series to obtain a Taylor polynomial, and evaluate it at x.

To find an approximation for $f(x)$, for a particular function f at a particular point x, to m decimal places (rule of thumb)

Use the method above to calculate approximations by using Taylor polynomials of degrees 1, 2, 3, and so on, until two successive different approximations agree to $m + 2$ decimal places.

Uniqueness of Taylor series

Let f be a function. If you can by any means find a series

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

that is equal to $f(x)$ for all x in some open interval containing a, then this series is the Taylor series about a for f, and hence it is the *only* series of this form that is equal to $f(x)$ for all x in that interval.

Manipulating Taylor series

You can obtain new Taylor series from standard Taylor series by:

- substituting for the variable
- adding, subtracting and multiplying Taylor series
- differentiating and integrating Taylor series term by term.

In each case, you can find an interval of validity for the new Taylor series from the known interval(s) of validity of the original Taylor series. In particular:

- A Taylor series about a point a for the sum, difference or product of two functions is valid for all values of x for which the Taylor series about a for both original functions are valid, and possibly for a larger interval of values.
- If an interval of validity for a Taylor series about a point a for a function f is an open interval, then it is also an interval of validity for the Taylor series about a for f' and for any antiderivative of f.

Unit 12 Complex numbers

Complex numbers and their arithmetic

The number i is defined to have the property $i^2 = -1$.

A **complex number** is a number of the form $a + bi$, where a and b are real numbers.

If $z = a + bi$, then

- the **real part** of z, denoted by $\operatorname{Re}(z)$, is a
- the **imaginary part** of z, denoted by $\operatorname{Im}(z)$, is b
- the **complex conjugate** of z, denoted by $\overline{z}$, is $a - bi$.

To manipulate complex numbers, use the usual rules of algebra and the fact that $i^2 = -1$.

To simplify a quotient of complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator.

Square roots of a negative real number

If d is a positive real number, then the square roots of $-d$ are $\pm i\sqrt{d}$.

Properties of complex conjugates

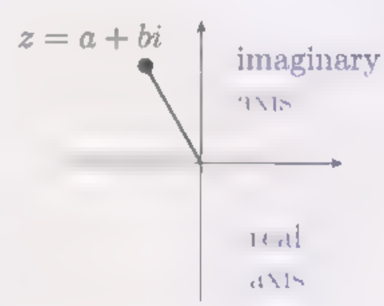
$$\overline{z+w} = \overline{z} + \overline{w} \qquad \overline{z-w} = \overline{z} - \overline{w}$$

$$\overline{zw} = \overline{z}\,\overline{w} \qquad \overline{z/w} = \overline{z}/\overline{w}$$

$$z\,\overline{z} = a^2 + b^2, \quad \text{where } z = a + bi$$

Modulus and argument

In the **complex plane** (the **Argand diagram**), the complex number $a + bi$ is represented by the point (a, b).

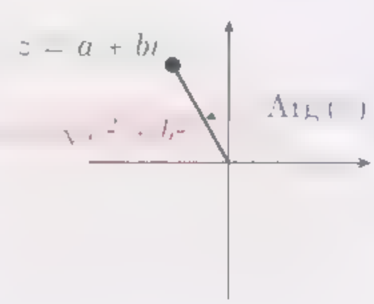


If $z = a + bi$, then:

- The **modulus** (**absolute value**) of z, denoted by $|z|$, is its distance from the origin, given by

$$|z| = \sqrt{a^2 + b^2}.$$

- An **argument** of z is an angle in radians measured anticlockwise from the positive real axis to the line between the origin and z.
- The **principal argument** of z, denoted by $\operatorname{Arg}(z)$, is the argument of z that lies in the interval $(-\pi, \pi]$.



The number 0 does not have an argument or a principal argument.

Properties involving modulus

$$z\overline{z} = |z|^2 \qquad |zw| = |z||w| \qquad \left|\frac{z}{w}\right| = \frac{|z|}{|w|} \qquad \frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

Forms of complex numbers

Cartesian, polar and exponential form

A complex number z can be written in any of the following three forms.

- **Cartesian form**: $z = a + bi$, where a and b are real numbers.
- **Polar form**: $z = r(\cos\theta + i\sin\theta)$, where r is the modulus of z and θ is an argument of z.
- **Exponential form**: $z = re^{i\theta}$, where r is the modulus of z and θ is an argument of z.

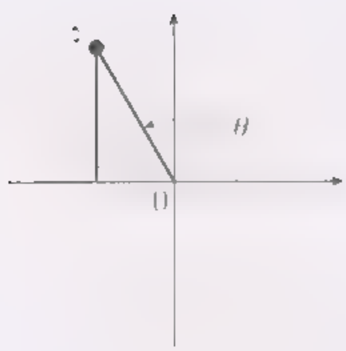
The expression $e^{i\theta}$ is defined by **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

To convert from Cartesian form to polar or exponential form

Given a non-zero complex number $z = a + bi$, proceed as follows.

1. Find the modulus r, using $r = |z| = \sqrt{a^2 + b^2}$.
2. Find the principal argument θ:



   (a) Mark z in roughly the right position in the complex plane, draw the line from 0 to z, and mark and label the principal argument θ.

   (b) If z lies on one of the axes, then use your diagram to find the principal argument θ. Otherwise carry out steps (c) to (e).

   (c) Label the acute angle between the real axis and the line from 0 to z as ϕ.

   (d) Draw a line from z perpendicular to the real axis to form a right-angled triangle, and mark the lengths of the horizontal and vertical sides.

   (e) Use the triangle to work out the angle ϕ and hence the principal argument θ.

3. Write z in polar form $z = r(\cos\theta + i\sin\theta)$ or exponential form $z = re^{i\theta}$, as required.

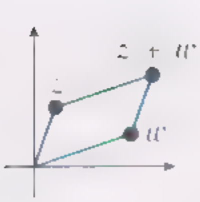
To convert from polar or exponential form to Cartesian form

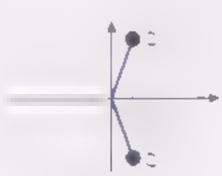
Write the complex number in polar form $z = r(\cos\theta + i\sin\theta)$, evaluate $\cos\theta$ and $\sin\theta$, and multiply out the brackets.

Euler's equation

$$e^{i\pi} + 1 = 0$$

Geometry of complex numbers





- Adding two complex numbers is similar to adding two vectors: a **parallelogram law** holds.
- Multiplying a complex number by a real number is similar to multiplying a vector by a scalar.
- Taking the complex conjugate of a complex number reflects it in the real axis.

Working with complex numbers in polar or exponential form

Complex conjugates

Polar form If $z = r(\cos\theta + i\sin\theta)$, then $\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$.

Exponential form If $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$.

Products and quotients

Polar form Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. Then

$$zw = rs\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big)$$
$$\frac{z}{w} = \frac{r}{s}\big(\cos(\theta-\phi) + i\sin(\theta-\phi)\big).$$

Exponential form Let $z = re^{i\theta}$ and $w = se^{i\phi}$. Then

$$zw = rs\,e^{i(\theta+\phi)}$$
$$\frac{z}{w} = \frac{r}{s}\,e^{i(\theta-\phi)}.$$

De Moivre's formula (formula for powers)

Polar form Let $z = r(\cos\theta + i\sin\theta)$. Then, for any integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Exponential form Let $z = re^{i\theta}$. Then, for any integer n,

$$z^n = r^n e^{in\theta}.$$

Equality

If $r(\cos\theta + i\sin\theta) = s(\cos\phi + i\sin\phi)$ or, equivalently, $re^{i\theta} = se^{i\phi}$, then

$$r = s \quad \text{and} \quad \theta = \phi + 2m\pi \text{ for some integer } m.$$

To work out a power of a complex number in Cartesian form

1. Write z in polar form, $z = r(\cos\theta + i\sin\theta)$.
2. Apply de Moivre's formula.
3. Convert the result back to Cartesian form.

(For small positive values of n this strategy is unnecessary – use the usual arithmetic of complex numbers.)

Some formulas useful for finding trigonometric identities

Special case of de Moivre's formula

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

Formulas deduced from Euler's formula

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Polynomial equations

A **polynomial equation** is an equation of the form 'polynomial expression = 0', where the polynomial expression has degree at least 1.

A **complex solution** of a polynomial equation is a solution of the equation that is a complex number. (It may be a real number, since every real number is also a complex number.)

To find the complex solutions of a quadratic equation

Simplify it, then use one of the following methods.

Completing the square

As on page 22, but if necessary use the fact that if d is a positive real number, then the two square roots of $-d$ are $\pm i\sqrt{d}$.

The quadratic formula

As on page 22, the solutions of the quadratic equation $az^2 + bz + c = 0$, where a, b and c are real numbers, are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac$ is negative, say $b^2 - 4ac = -d$ where d is positive, then $\pm\sqrt{b^2 - 4ac}$ means $\pm i\sqrt{d}$ (the two square roots of the negative number $b^2 - 4ac$).

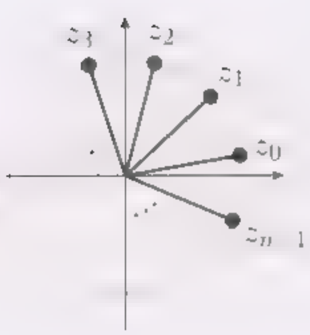
Roots of complex numbers

An **nth root** of a complex number a is a solution z of the equation $z^n = a$, where a is a complex number and n is a positive integer.

An **nth root of unity** is a solution z of the equation $z^n = 1$, where n is a positive integer.

To find the complex solutions of the equation , where

(There are n solutions, equally spaced on a circle centred at 0.)



1. Write the unknown z in polar form, in terms of an unknown modulus r and an unknown argument θ, and write the number a in polar form.
2. Substitute the polar forms of z and a into the equation, and apply de Moivre's formula to find the polar form of the left-hand side.
3. Compare moduli to find the value of r.
4. Compare arguments to find n successive possible values of θ.
5. Hence write down the n possible values of z.

It is usually convenient to use arguments in the interval $[0, 2\pi)$.

The fundamental theorem of algebra

Every polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ of degree $n \geq 1$ has a factorisation $a_n(z - z_1)(z - z_2)\cdots(z - z_n)$, where $z_1, z_2, \ldots, z_n$ are complex numbers, some of which may be equal to others.

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